Universitat de les Illes Balears

DEA WP no. 95
Working Paper Series

Exploiting homogeneity in games with non-homogeneous revenue functions

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July 2022

Departament d'Economia Aplicada

Noname manuscript No.
(will be inserted by the editor)

Exploiting homogeneity in games with non-homogeneous revenue functions

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Received: date / Accepted: date

Abstract We exploit the properties of homogeneous functions to characterize the symmetric pure-strategy Nash equilibria of n-player games in which each player's revenue function is not homogeneous but it can be decomposed into the sum of homogeneous functions with different degrees of homogeneity. These features are met in a wide range of games, including contests, imperfect competition, or public good games. Our results aim to contribute an easy checklist for finding the symmetric pure-strategy Nash equilibria. We apply these results to three examples.

Keywords equilibrium characterization · homogeneous functions' properties · non-homogeneous revenue function

JEL classification Q58 · H23 · D72

Statements and Declarations

The authors declare that they have no conflict of interest.

We wish to thank Daniel Cardona, José Vicente Pérez and József Sákovics for their useful comments. We acknowledge financial support from the Spanish Ministerio de Ciencia y Educación and Ministerio de Universidades (Agencia Estatal de Investigación) through the project PID2019-107833GB-I00/MCIN/AEI/10.13039/501100011033. A previous version of this article circulated under the title "Homogeneous Functions and Nash Equilibria".

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1 Introduction

Homogeneous functions frequently appear in Economic Theory models. For example, in consumption theory the indirect utility function is homogeneous of degree zero in prices and income and the expenditure function is homogeneous of degree one in prices; in general equilibrium, the excess demand function is homogeneous of degree zero.

The properties of homogeneous functions have been exploited by Malueg and Yates (2006) to show sufficient conditions for the existence of a unique symmetric pure-strategy Nash equilibrium in rent-seeking contests. More specifically, in their setting, the players' revenue (utility net of cost) is described by a unique homogeneous function of degree zero. Ferrarese (2021) generalizes those results by allowing such a function to exhibit any degree of homogeneity. This allowed the author to extend the set of possible applications beyond the games considered by Malueg and Yates (2006). For instance, the results in Ferrarese (2021) can be applied to contests with an endogenous prize valuation or to Cournot games with a non-linear inverse demand.

This paper aims to extend the characterization of symmetric pure-strategy Nash equilibria (henceforth, SPNE) to a wider set of games. Specifically, we show that the properties of homogeneous functions can also be exploited to characterize the SPNE in games where the players' revenue function is not (necessarily) homogeneous but it can be decomposed into the sum of homogeneous functions with (possibly) different degrees of homogeneity.[1] A simple example may help the reader. Let us consider a linear, symmetric, homogeneous-good Cournot game between two firms i and j. The inverse demand is $p_i(q_i,q_j) = \alpha - q_i - q_j$, $\alpha > 0$ and q_i and q_j represent the firms' quantities. Thus, firm i's revenue is $\mathscr{R}_i(q_i,q_j) = (\alpha - q_i - q_j)\,q_i$. Notice that $\mathscr{R}_i$ is not homogeneous. However, one can decompose it as $\mathscr{R}_i(q_i,q_j) = \alpha q_i - \left(q_i q_j - q_i^2\right)$ where the first term is a homogeneous function of degree one, and the second term is a homogeneous function of degree two.

Hence, the results presented in this paper apply not only to the games covered by Ferrarese (2021) and Malueg and Yates (2006) but to others, as well. In particular, we show how our results can be used to characterize the equilibria of games classified within several families of games such as, public good, contests, and imperfect competition games.

The extra generality of our setting comes at a cost: The set of symmetric equilibrium candidates cannot be reduced to a singleton and there is not a closed form solution for them. However, because of homogeneity, (i) we characterize the equilibrium candidates as the roots of a generalized polynomial whose real coefficients are represented by the partial derivatives of the component functions of a player's payoff evaluated at a specific point. This contributes a shortcut for economic theory researchers to identify the maximum number of interior equilibrium candidates, as they can use the mathematical results for counting the positive zeros of generalized polynomials (Jameson, 2006). Moreover, (ii) we provide sufficient conditions for each of these candidates to be a SPNE, and (iii) we show under which conditions SPNE (if any) must be interior. Finally, we apply results (i)-(iii) to characterize the equilibria of three examples pertaining to the families of games commented above.

The paper is structured as follows: in Section 2, we briefly present some useful properties of homogeneous functions; in Section 3, we present the model; Section 4 is devoted to the equilibrium analysis; in Section 5, we present applications and Section 6 concludes. All proofs are in the Appendix.

2 Preliminaries: Some results on homogeneous functions

Throughout the paper, we will make use of some properties of homogeneous functions. A real valued function $f : \mathbb{R}^I \to \mathbb{R}$ is homogeneous of degree α in $\mathbf{x} \equiv (x_1, x_2, ..., x_I)$ if $f(t\mathbf{x}) = t^\alpha f(\mathbf{x}), \forall t > 0$. This

[1] The sum of homogeneous functions is not homogeneous unless all the addends share the same degree of homogeneity.

implies that, given a vector with I identical entries $\check{\mathbf{x}} \equiv (x,x,...,x)$, $f(\check{\mathbf{x}}) = x^{\alpha} f(\mathbf{1})$, where $\mathbf{1}$ is an I-dimensional vector of ones. Furthermore, the two following remarks will be useful:

Remark 1 Let $f : \mathbb{R}^I \to \mathbb{R}$ be differentiable and homogeneous of degree α. Then the n-th derivative is homogeneous of degree $\alpha - n$.

Proof Omitted

It follows that for the n-th partial derivative, $\frac{\partial^n f}{\partial x_i^n}(\mathbf{x}) = x^{\alpha-n} \frac{\partial^n f}{\partial x_i^n}(\mathbf{1})$. The last remark is the following:

Remark 2 The set of homogeneous functions is closed with respect to the operation of multiplication and is not (in general) closed with respect to the operation of sum. The sum of homogeneous functions is itself homogeneous if all terms in the sum exhibit the same degree of homogeneity.

Proof Omitted

3 The Model

Let $\mathscr{I} = \{1,2,...,I\}$ be the index set of players, each selecting an $x_i \in [0,+\infty)$ simultaneously and non-cooperatively. Let $\mathbf{x} \equiv (x_1,..,x_I)$ be an I-dimensional vector of strategies selected by all players. Player i's payoff is given by:

$$u_i(\mathbf{x}) = \mathscr{R}_i(\mathbf{x}) - \mathscr{C}_i(\mathbf{x}).$$

Let the cost function $\mathscr{C}_i : \mathbb{R}_+^I \to \mathbb{R}_+$ be a homogeneous function of degree $s \in \mathbb{R}_{++}$ with $\mathscr{C}_i(0,\mathbf{x}_{-i}) = 0 \; \forall \; \mathbf{x}_{-i} \in \mathbb{R}_+^{I-1}$, $\frac{\partial \mathscr{C}_i}{\partial x_i} > 0$ and $\frac{\partial^2 \mathscr{C}_i}{\partial x_i^2} \geq 0$.[2] The revenue function is $\mathscr{R}_i(\mathbf{x}) = \sum_{h \in \mathscr{H}} f_{i,h}(\mathbf{x})$, where each $f_{i,h}$ is a homogeneous function of degree $\alpha_h \in \mathbb{R}$, with $f_h(\mathbf{0}) \geq 0$, $\forall h \in \mathscr{H}$, and $\mathscr{H} = \{\underline{h}, \underline{h}+1, ..., \overline{h}\}$ is the index set of these functions holding the following order: $\alpha_{\underline{h}} < \alpha_{\underline{h}+1} < ... < \alpha_{\overline{h}}$. Furthermore, $\mathscr{C}_i$ and $f_{i,h} \; \forall \; h \in \mathscr{H}$ are continuous and differentiable over $\mathbb{R}_{++}^I$ and possibly over $\mathbb{R}_+^I$.[3] Hence, this setting allows for the payoff function to have a point of discontinuity at the origin.[4] Let $\mathscr{H}_d \subseteq \mathscr{H}$ be the index set of functions with a discontinuity at the origin. Additionally, let $\mathscr{H}_{-d} \subseteq \mathscr{H}$ and $\mathscr{H}_{-d_0} \subseteq \mathscr{H}$ be the index set of continuous functions over $\mathbb{R}_+^I$ such that $f_h(\mathbf{0}) > 0$ and $f_h(\mathbf{0}) = 0$, respectively.

Let $\mathscr{G}$ denote the family of games with the above features. Notice that the set of games covered by Ferrarese (2021) is a subset of $\mathscr{G}$, as it considers cases in which $\mathscr{H}$ has a unique element h, hence the revenue function is homogeneous of degree $\alpha_h \in \mathbb{R}$. The set of games analyzed by Malueg and Yates (2006) is even more restricted as the revenue function is homogeneous of degree zero, hence $\mathscr{H}$ has a unique element h such that $\alpha_h = 0$.

A SPNE of $g \in \mathscr{G}$ is a strategy profile $\mathbf{x}^* = (x_1^*, x_2^*, ..., x_I^*) \in \mathbb{R}_+^I$ with $x_i^* = x_j^*, \forall i \neq j$, such that:

$$\sum_{h \in \mathscr{H}} \left(f_{i,h}(\mathbf{x}^*) - f_{i,h}(x_i, \mathbf{x}_{-i}^*) \right) - \left(\mathscr{C}_i(\mathbf{x}^*) - \mathscr{C}_i(x_i, \mathbf{x}_{-i}^*) \right) \geq 0, \forall x_i \neq x_i^* \text{ and } \forall i \in \mathscr{I}.$$

4 Equilibrium Analysis

We now exploit the properties of homogeneous functions to *(i)* characterize the interior SPNE candidates, *(ii)* show sufficient conditions for each candidate to be a SPNE, and *(iii)* determine the conditions that exclude the null vector as a SPNE. The first result is the following:

[2] $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$.

[3] $\mathbf{0}$ is an I-dimensional vector of zeros.

[4] A contest success function typically satisfies this feature.

Proposition 1 *If an interior SPNE of $g \in \mathscr{G}$ exists then the equilibrium strategy is a zero of the generalized polynomial:*

$$\mathscr{P}(x) = \sum_{h \in \mathscr{H}} \frac{\partial f_{i,h}}{\partial x_i}(\mathbf{1})\, x^{\alpha_h - 1} - \frac{\partial \mathscr{C}_i}{\partial x_i}(\mathbf{1})\, x^{s-1},$$

with $\deg(\mathscr{P}) = \max\{\alpha_{\bar{h}} - 1, s-1\}$.

When $\#\mathscr{H} = 1$, a common feature in Malueg and Yates (2006) and Ferrarese (2021), a closed form characterization of a unique symmetric equilibrium candidate can be obtained as a zero of $\mathscr{P}(x)$.[5] However, for those $g \in \mathscr{G}$ for which $\#\mathscr{H} > 1$, this is no longer the case. Instead, the properties of homogeneous functions allow us to characterize the equilibrium candidates as the zeros of the generalized polynomial $\mathscr{P}(x)$, whose coefficients are the partial derivatives of the family of functions $f_{i,h}$ and $\mathscr{C}_i$ with respect to x_i, evaluated at a vector of ones.

Due to the characterization of the SPNE candidates presented in the previous Proposition, we can take advantage of the results regarding the number of positive zeros of polynomials to offer a shortcut to identify the maximum number of interior SPNE candidates. A known property regarding the number of positive roots of a polynomial is the Descartes' rule of sign, stating that the number of positive roots is no larger than the number of sign changes of its coefficients. As in our case the exponents of $\mathscr{P}(x)$ can be non-integers, we should look at the generalization of the Descartes' rule to generalized polynomials due to Jameson (2006):

Theorem 1 *Jameson (2006). Let*

$$w(x) = \sum_{j=1}^{n} a_j x^{p_j}$$

with $x \in \mathbb{R}_{++}$, $p_j \in \mathbb{R}$, and $p_1 > p_2 > \ldots > p_n$. Let $z(w)$ be the number of zeros of $w(x)$. Then $z(w)$ is no greater than the number of sign changes in $w(x)$.

Applying this rule to our setting, we can easily know the maximum number of equilibrium candidates by counting the number of sign changes in the ordered version of $\mathscr{P}(x)$, say $\hat{\mathscr{P}}(x)$, in which the exponents of x ($s-1$ and $\alpha_h - 1$ for all $h \in \mathscr{H}$) appear in an ordered way.[6] Hence, in order to compute the maximum number of interior SPNE candidates, one just need to analyze the partial derivatives, evaluated at a vector of ones, of the cost function and all the component functions of $\mathscr{R}_i(\mathbf{x})$. An immediate consequence of the previous result is that in case of a unique sign change, there is at most one interior SPNE candidate.

At this point, let $\bar{x}$ be a positive solution of $\mathscr{P}(x)$, so that $\bar{\mathbf{x}} \equiv (\bar{x}, \bar{x}, \ldots, \bar{x})$ is an interior SPNE candidate. The following guarantees that $\bar{\mathbf{x}}$ is, indeed, a SPNE.

Proposition 2 $\bar{\mathbf{x}}$ *is a SPNE of $g \in \mathscr{G}$ if the following two requirements are satisfied:*

I) $\mathscr{C}_i(\mathbf{1})^{-1} \sum_{h \in \mathscr{H}_d \cup \mathscr{H}_{-d_0}} \left(\frac{f_{i,h}(\mathbf{1}) - f_{i,h}(0, \mathbf{1}_{-i})}{\bar{x}^{s-\alpha_h}} \right) > 1$*;*

II) $\sum_{h \in \mathscr{H}_d \cup \mathscr{H}_{-d_0}} \bar{x}^{\alpha_h - s} \frac{\partial^2 f_{i,h}}{\partial x_i^2} \left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i} \right) - \frac{\partial^2 \mathscr{C}_i}{\partial x_i^2} \left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i} \right) > 0, \forall x_i < \tilde{x}$*, for some $\tilde{x} \in [0, \bar{x})$, and negative otherwise.*

Requirement *I)* ensures that player i prefers $\bar{x}$ to 0 when all the remaining contenders are selecting $\bar{x}$ as well. This condition is needed to exclude the possibility of $\bar{x}$ being just a local maximum of $u_i\left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i}\right)$, as 0 is at the corner of the players' strategy set. Requirement *II)* implies that $u_i\left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i}\right)$ is quasiconcave

[5] In this case, the unique symmetric candidate is $\bar{x} = \left(\frac{\partial \mathscr{C}_i}{\partial x_i}(\mathbf{1}) / \frac{\partial \mathscr{R}_i}{\partial x_i}(\mathbf{1}) \right)^{\alpha - s}$.

[6] Notice also that $s-1$ may coincide with $\alpha_h - 1$ for at most one $h \in \mathscr{H}$.

and that the equilibrium candidate $\bar{x}$ lies on its concave region. These two conditions ensure that $\bar{x} = \underset{x_i \in R_+}{arg\,max}\, u_i\left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i}\right)$, *i.e.* $\bar{x}$ is the best-response of player i to $\bar{x}_{-i}$.[7]

We can also exploit homogeneity to provide insights on whether a game $g \in \mathscr{G}$ admits the null vector as a symmetric equilibrium. This might be a relevant analysis in some games (like public good games) where the null vector is a potential equilibrium. In this regard, we have the following result:

Proposition 3 *The null vector is not a SPNE of $g \in \mathscr{G}$ if and only if one of the following disjoint sets of conditions holds*

a) $\alpha_{\underline{h}} < 0$ *and* $f_{i,\underline{h}}(1, \mathbf{0}_{-i}) > 0$ *;*
b) $\alpha_{\underline{h}} = 0$*, and* $f_{i,\underline{h}}(1, \mathbf{0}_{-i}) - f_{i,\underline{h}}(\mathbf{0}) > 0$*;*
c) $s > \alpha_{\underline{h}} > 0$*,* $f_{i,h}(\mathbf{0}) = 0 \forall h \in \mathscr{H}_d$*, and* $f_{i,\underline{h}}(1, \mathbf{0}_{-i}) > 0$*;*
d) $s = \alpha_{\underline{h}} > 0$*,* $f_{i,h}(\mathbf{0}) = 0 \forall h \in \mathscr{H}_d$*, and* $\mathscr{C}_i(1, \mathbf{0}_{-i})^{-1} f_{i,\underline{h}}(1, \mathbf{0}_{-i}) > 1$.

These conditions are necessary and sufficient for excluding the null vector from the set of SPNE, or equivalently, for having an $\hat{x} > 0$ such that $u_i(x_i, \mathbf{0}_{-i}) > u_i(\mathbf{0})$ for all $x_i \in (0, \hat{x})$. As it can be seen, functions' homogeneity is exploited here to express these conditions in terms of the degree of homogeneity of the cost function s and the lowest degree of homogeneity in $\mathscr{H}$, $\alpha_{\underline{h}}$. Notice that, in order to exclude $\mathbf{0}$ from the set of SPNE, the smallest degree of homogeneity of a component function of $\mathscr{R}_i$, $\alpha_{\underline{h}}$, cannot be higher than the degree of homogeneity of the cost function s.

Point b) nests Malueg and Yates (2006), where the unique component function in $\mathscr{R}_i$ is the product between the Tullock's contest success function and the exogenous value of the prize V. This function is homogeneous of degree zero, discontinuous at the origin, takes the value $\frac{V}{I} > 0$ at such point and the value V at $(x_i, \mathbf{x}_{-i}) = (1, \mathbf{0})$. However, condition b) also applies to other games with a non-homogenous revenue function, as illustrated by our third example in the next section.

Point c) applies to the abatement game described below. Point d) applies to our second example, a Cournot game with differentiated products.

5 Applications

We now provide three applications: the first one is a special public good (bad) game, the second one is a standard game of firm competition, and the third example is a contest game.

Example 1: *Abatement games*

For this application we rely on Barrett (1994), where each i out of I countries emits a quantity of a pollutant $x_i \in \mathbb{R}_+$ damaging a shared natural resource. Country i's revenue is:

$$\mathscr{R}_i(x_i, \mathbf{x}_{-i}) = \psi\left(\sum_{j=1}^{I} x_j\right) - \sigma\left(\sum_{j=1}^{I} x_j\right)^2,$$

with $\psi > 0$ and $\sigma > 0$, which is not homogeneous. However, one can write that $\mathscr{R}_i = f_{i,1}(x_i, \mathbf{x}_{-i}) + f_{i,2}(x_i, \mathbf{x}_{-i})$, where $f_{i,1}(x_i, \mathbf{x}_{-i}) = \psi\left(\sum_{j=1}^{I} x_j\right)$ and $f_{i,2}(x_i, \mathbf{x}_{-i}) = -\sigma\left(\sum_{j=1}^{I} x_j\right)^2$ are both homogeneous. Since player i's cost function is $\mathscr{C}_i(x_i) = cx_i^2$ with $c > 0$, then her payoff is:

$$u_i(x_i, \mathbf{x}_{-i}) = \psi\left(\sum_{j=1}^{I} x_j\right) - \sigma\left(\sum_{j=1}^{I} x_j\right)^2 - cx_i^2,$$

[7] When these conditions are not satisfied then each $\bar{x}$ can be either a global or a local maximum/minimum of $u_i(x_i, \bar{\mathbf{x}}_{-i})$. Hence, checking that $\bar{x}$ is a global maximum requires further analysis.

and according to the notation of the paper $\alpha_1 = \alpha_{\underline{h}} = 1$ and $\alpha_2 = \alpha_{\overline{h}} = 2 = s$.[8] By symmetry, we focus on the problem of a representative player i only. As:

$$\frac{\partial f_{i,1}}{\partial x_i}(\mathbf{1}) = \psi;$$
$$\frac{\partial f_{i,2}}{\partial x_i}(\mathbf{1}) = -2\sigma I;$$
$$\frac{\partial \mathscr{C}_i}{\partial x_i}(1) = 2c$$

then, if an interior SPNE exists, it is a root of the degree 1 polynomial:

$$\mathscr{P}(x) = \psi - x(2\sigma I - 2c).$$

Given that $\psi > 0$, we apply the Descartes' rule of sign to conclude that the number of positive zeros of the above polynomial is at most one when $\sigma > c/I$. This positive root is given by:

$$\bar{x} = \frac{\psi}{2\,(I\sigma - c)}.$$

Hence, $\bar{\mathbf{x}} = (\bar{x}, ..., \bar{x})$ is the unique interior SPNE candidate. Since:

$$f_{i,1}(\mathbf{1}) = \psi I;$$
$$f_{i,2}(\mathbf{1}) = -\sigma I^2;$$
$$f_{i,1}(0, \mathbf{1}_{-i}) = \psi(I-1);$$
$$f_{i,2}(0, \mathbf{1}_{-i}) = -\sigma(I-1)^2;$$

condition I) in Proposition 2 is:

$$2(\sigma I - c) + \sigma\,(1 - 2I) > c,$$

which requires $\sigma > 3c$. Notice that, since $I \geq 2$ and $c > 0$, $\max\{3c, c/I\} = 3c$, this condition is more stringent that the previous one. It only remains to check whether $x_i = \bar{x}_i$ is the best reply to $\mathbf{x}_{-i} = (\bar{x}, \bar{x}, ..., \bar{x})$. Since:

$$\frac{\partial^2 f_{i,1}}{\partial x_i^2}(x_i) = 0;$$
$$\frac{\partial^2 f_{i,2}}{\partial x_i^2}(x_i) = -2\sigma;$$
$$\frac{\partial^2 \mathscr{C}_i}{\partial x_i^2}(x_i) = 2c,$$

condition II) in Proposition 2 needs to be check for $-2\,(\sigma + c)$, which is negative. Hence the unique interior SPNE is $(x_1^*, x_2^*, ..., x_I^*) = (\bar{x}, \bar{x}, ..., \bar{x})$ if $\sigma > 3c$. Finally, since $\mathscr{H}_d = \emptyset$ and $s > \alpha_{\underline{h}} > 0$, then the null vector is not an equilibrium if $f_{i,1}(1, \mathbf{0}_{-i}) - f_{i,1}(\mathbf{0}) = \psi > 0$, which is clearly true. Thus, the above interior equilibrium is also the unique symmetric Nash equilibrium of the abatement game.

Example 2*: A linear horizontally differentiated Cournot duopoly*

[8] The abatement game is a special case of *quadratic games* as defined in Dokka *et al.* (2021), a class of n-player games in which $u_i(x_i, \mathbf{x}_{-i}) = \psi\left(\sum_{i=1}^{I} x_i\right) + \sigma \sum_{i=1}^{I} \sum_{j=1}^{I} x_i x_j + t x_i^2$.

Two firms $i = \{1,2\}$ face the linear inverse demand $p_i = \gamma - \beta_1 q_i - \beta_2 q_j$, where $\gamma > 0$ is a measure of market extension, and $\beta_1 > 0$ and β_2 are two parameters capturing the degree of differentiation between goods: when $\beta_2 > 0$ the goods are substitutes, when $\beta_2 = 0$ the goods are independent,when $\beta_2 < 0$ the goods are complements.[9] We make the typical assumption that own effects dominate cross effects, namely $\beta_1 >\mid \beta_2 \mid$. Each firm selects a quantity $q_i \in \mathbb{R}_+$ and produces with the total cost function $\mathscr{C}(q_i) = cq_i$, where $c > 0$. Thus, firm i's revenue is:

$$\mathscr{R}_i(q_i,q_j) = (\gamma - \beta_1 q_i - \beta_2 q_j)\, q_i,$$

which is not homogeneous. However, one can write that $\mathscr{R}_i = f_{i,1}(q_i) + f_{i,2}(q_i,q_j)$, where $f_{i,1}(q_i) = \gamma q_i$ and $f_{i,2}(q_i,q_j) = -\beta_1 q_i^2 - \beta_2 q_i q_j$ are both homogeneous. Since player i's payoff is:

$$\pi_i(q_i,q_j) = (\gamma - \beta_1 q_i - \beta_2 q_j)\, q_i - cq_i,$$

according to the notation of the paper $\alpha_1 = \alpha_{\underline{h}} = s = 1$ and $\alpha_2 = \alpha_{\overline{h}} = 2$. By symmetry, we focus on the problem of firm 1 only. As:

$$\frac{\partial f_{1,1}}{\partial q_1}(1,1) = \gamma;$$
$$\frac{\partial f_{2,1}}{\partial q_1}(1,1) = -(2\beta_1 + \beta_2);$$
$$\frac{\partial \mathscr{C}_1}{\partial q_1}(1,1) = c$$

then, if a SPNE exists, it is a root of the degree 1 polynomial:

$$\mathscr{P}(q_1) = (\gamma - c) - q_1(2\beta_1 + \beta_2).$$

Given that $2\beta_1 + \beta_2 > 0$, we apply the Descartes' rule of sign to conclude that the number of positive zeros of the above polynomial is at most one when $\gamma > c$. This positive root is given by:

$$\bar{q} = \frac{\gamma - c}{2\beta_1 + \beta_2}.$$

Hence, $\bar{\mathbf{q}} = (\bar{q}, ..., \bar{q})$ is the unique interior SPNE candidate. Since:

$$f_{1,1}(1,1) = \gamma;$$
$$f_{2,1}(1,1) = -(2\beta_1 + \beta_2);$$
$$f_{1,1}(0,1) = f_{2,1}(0,1) = 0,$$

condition I) in Proposition 2 is:

$$(\gamma - c)\left(\frac{\beta_1}{2\beta_1 + \beta_2}\right) > 0,$$

which again requires $\gamma > c$.

[9] Although it is known that this game admits a unique symmetric equilibrium, this has not been shown through the use of homogeneous functions. It is also worth noting that the paper incorporates less straightforward cases such as Cournot games with non-linear inverse demand and non-constant returns to scale (Ferrarese, 2021).

It only remains to check whether $q_1 = \bar{q}$ is the best reply to $q_2 = \bar{q}$. Since:

$$\frac{\partial^2 f_{1,1}}{\partial q_1^2}(q_1,1) = 0;$$
$$\frac{\partial^2 f_{2,1}}{\partial q_1^2}(q_1,1) = -2\beta_1;$$
$$\frac{\partial^2 \mathscr{C}_1}{\partial q_1^2}(q_1,1) = 0,$$

condition II) in Proposition 2 needs to be check for $-2\bar{q}\beta_1$, which is negative. Hence the unique interior SPNE is $(q_1^*, q_2^*) = (\bar{q}, \bar{q})$ if $\gamma > c$.[10] Finally, since $\mathscr{H}_d = \emptyset$ and $s = \alpha_{\underline{h}} > 0$, then the null vector is not an equilibrium if $f_1(1,0)/\mathscr{C}(1,0) = \gamma/c > 1$. So that, the same condition $\gamma > c$ ensures that $(\bar{q}, \bar{q})$ is the only SPNE of the game.[11]

Example 3: *A generalized Tullock contest à la Chowdhury and Sheremeta (2011a,b)*

Two risk-neutral players $i = \{1,2\}$ exert an irreversible effort $e_i \in \mathbb{R}_+$. Contingent upon winning or losing, each player obtains a prize $W > 0$ and $L \in \mathbb{R}$, with $W > L$, respectively. Player i's revenue is:

$$\mathscr{R}_i(e_i,e_j) = \varphi_i(\mathbf{e})\,(W + \beta_1 e_j) + (1-\varphi_i(\mathbf{e}))\,(L + \beta_2 e_j),$$

where

$$\varphi_i(\mathbf{e}) = \begin{cases} \frac{e_i}{e_i+e_j} & \text{if } \mathbf{e} \neq \mathbf{0} \\ \frac{1}{2} & \text{if } \mathbf{e} = \mathbf{0} \end{cases}$$

is the contest success function which captures player i's probability of winning the contest or her prize share, β_1 and β_2 are two spillover parameters. Notice that $\mathscr{R}_i(e_i,e_j)$ is not homogeneous. However, one can write that $\mathscr{R}_i(e_i,e_j) = f_{i,1}(e_i,e_j) + f_{i,2}(e_i,e_j)$, where $f_{i,1}(e_i,e_j) = \varphi_i(\mathbf{e})(W-L) + L$ and $f_{i,2}(e_i,e_j) = \beta_2 e_j + e_j \varphi_i(\mathbf{e})(\beta_1 - \beta_2)$, which are both homogeneous. Since player i's cost function is $\mathscr{C}_i(e_i,e_j) = \theta_2 e_i + \varphi_i(\mathbf{e})(\theta_1 - \theta_2)e_i$, where in order to ensure that effort is costly, $\theta_1 > 0$, $\theta_2 \geq 0$, then her utility is given by:

$$u_i(\mathbf{e}) = \varphi_i(\mathbf{e})\,(W + \beta_1 e_j) + (1-\varphi_i(\mathbf{e}))\,(L + \beta_2 e_j) - \theta_2 e_i + \varphi_i(\mathbf{e})(\theta_1 - \theta_2)e_i.$$

Consistent with the notation of the paper $\alpha_1 = \alpha_{\underline{h}} = 0$ and $\alpha_2 = \alpha_{\overline{h}} = s = 1$. By symmetry, we focus on the problem of contender 1 only. As:

$$\frac{\partial f_{1,1}}{\partial e_1}(1,1) = \frac{W-L}{4};$$
$$\frac{\partial f_{2,1}}{\partial e_1}(1,1) = \frac{\beta_1 - \beta_2}{4};$$
$$\frac{\partial \mathscr{C}_1}{\partial e_1}(1,1) = \theta_2 + \frac{3(\theta_1 - \theta_2)}{4} = \frac{3\theta_1 + \theta_2}{4},$$

if a SPNE exists, it is a root of the degree 1 polynomial:

$$\mathscr{P}(e_1) = \frac{W-L}{4} + e_1\left(\frac{\beta_1 - \beta_2}{4} - \frac{3\theta_1 + \theta_2}{4}\right).$$

[10] This application can be easily extended to $I > 2$ players.

[11] Another important case is the one where firms adopt a decreasing return to scale technology as in Szidarovszky and Yakowitz (1977). With quadratic costs, for instance, $s > \alpha_{\underline{h}} > 0$, so that according to point *c)* and being $\mathscr{H}_d = \emptyset$, the weaker condition $f_{i,1}(1,\mathbf{0}_{-i}) = \gamma > 0$ applies.

Given that $W > L$, we apply the Descartes' rule of sign to conclude that the number of positive zeros of the above polynomial is at most one when $\beta_1 - \beta_2 < 3\theta_1 + \theta_2$. This positive root is given by:

$$\bar{e} = \frac{W-L}{(3\theta_1 + \theta_2) - (\beta_1 - \beta_2)}.$$

Hence, $\bar{\mathbf{e}} = (\bar{e}, ..., \bar{e})$ is the unique interior SPNE candidate.

As:

$$f_{1,1}(1,1) - f_1(0,1) = \frac{W-L}{2};$$
$$f_{2,1}(1,1) - f_2(0,1) = \frac{\beta_1 - \beta_2}{2};$$
$$\mathscr{C}_1(1,1) = \frac{\theta_1 + \theta_2}{2},$$

condition I) in Proposition 2 is:

$$\frac{(3\theta_1 + \theta_2) - (\beta_1 - \beta_2)}{2} + \frac{\beta_1 - \beta_2}{2} > \frac{\theta_1 + \theta_2}{2},$$

which is satisfied as $\theta_1 > 0$.[12]

It only remains to check whether $e_1 = \bar{e}$ is the best reply to $e_2 = \bar{e}$. Since:

$$\frac{\partial^2 f_{1,1}}{\partial e_1^2}(e_1, 1) = \frac{-2}{(1+e_1)^3}(W-L);$$
$$\frac{\partial^2 f_{2,1}}{\partial e_1^2}(e_1, 1) = \frac{-2}{(1+e_1)^3}(\beta_1 - \beta_2);$$
$$\frac{\partial^2 \mathscr{C}_1}{\partial e_1^2}(e_1, 1) = \frac{2}{(1+e_1)^3}(\theta_1 - \theta_2),$$

condition II) in Proposition 2 needs to be check for:

$$-\frac{2}{(1+\frac{e_1}{\bar{e}})^3}\left(\frac{W-L+\bar{e}(\beta_1 - \beta_2 + \theta_1 - \theta_2)}{\bar{e}}\right).$$

Substituting $\bar{e}$, we obtain that the previous condition simplifies to:

$$-\frac{8\theta_1}{(1+\frac{e_1}{\bar{e}})^3},$$

which is negative, so that the unique interior SPNE is $(e_1^*, e_2^*) = (\bar{e}, \bar{e})$ if $\beta_1 - \beta_2 < 3\theta_1 + \theta_2$. Finally, as $\alpha_{\underline{h}} = 0$ and $f_{1,1}(1,0) - f_{1,1}(0,0) = \frac{W-L}{2} > 0$, according to point b) in Proposition 3, the null vector cannot be an equilibrium, and the above interior equilibrium is the unique symmetric equilibrium of the generalized Tullock contest.

[12] Chowdhury and Sheremeta (2011a,b) mistakenly compare the equilibrium payoffs with the payoff of loosing which, in this case, is not equal to the payoff of deviating from the equilibrium by exerting a zero effort. This is the reason why our condition does not coincide with theirs.

6 Conclusions

We characterized the SPNE candidates of n-person games in which the players' revenue takes the form of a finite sum of homogeneous functions as the positives zeros of a generalized polynomial, whose coefficients are represented by the partial derivatives of the component functions of a player's payoff evaluated at a vector of ones. This permits to use the mathematical results for counting the number of positive zeros of this kind of polynomials to determine the maximum number of interior SPNE candidates. We also provided sufficient conditions for each candidate to be a SPNE and show the conditions for excluding the null vector as a SPNE. Since the set of homogeneous functions is not (generally) closed with respect to the operation of sum, our results extend the set of applications of the previous literature. We have shown three examples to illustrate some of these extra applications.

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Appendix

Proof (Proposition 1) First, by symmetry, we focus on the problem of a representative player i. In a symmetric vector of strategies with positive entries $\mathbf{x}$, player i's payoff is:

$$\sum_{h\in\mathscr{H}} f_{i,h}(\mathbf{x}) - \mathscr{C}_i(\mathbf{x}).$$

By homogeneity and Remark 1, we can write that $\frac{\partial f_{i,h}}{\partial x_i}(\mathbf{x}) = x^{\alpha_h-1}\frac{\partial f_{i,h}}{\partial x_i}(\mathbf{1})$ and $\frac{\partial \mathscr{C}_i}{\partial x_i}(\mathbf{x}) = x^{s-1}\frac{\partial \mathscr{C}_i}{\partial x_i}(\mathbf{1})$. Thus, the necessary first order conditions for an interior equilibrium $0 = \frac{\partial u_i}{\partial x_i}$ can be rewritten as:

$$0 = \sum_{h\in\mathscr{H}} x^{\alpha_h-1}\frac{\partial f_{i,h}}{\partial x_i}(\mathbf{1}) - x^{s-1}\frac{\partial \mathscr{C}_i}{\partial x_i}(\mathbf{1}).$$

Proof (Proposition 2) We first establish the following result:

Lemma 1 *If a function $f:\mathbb{R}^n_+ \to \mathbb{R}$ is continuous, homogeneous of degree α with $f(\mathbf{0}) = C > 0 \Rightarrow f$ is the constant function $f(x) = C$.*

Proof By homogeneity $f(t\mathbf{0}) = t^\alpha f(\mathbf{0})\ \forall t > 0$. Since $f(t\mathbf{0}) = f(\mathbf{0}) = C > 0$, then $\alpha = 0$. Since f is homogeneous of degree 0, then $f(t\mathbf{x}) = t^\alpha f(\mathbf{x}) = f(\mathbf{x})\ \forall \mathbf{x} \in \mathbb{R}^n_+$, namely for a given $\mathbf{x} \in \mathbb{R}^n_+$, f is constant along the line passing through $(0, t\mathbf{x})$. By continuity at the origin, then $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $| f(\mathbf{x}) - C | < \varepsilon\ \forall x \in \mathbb{R}^n_+$ with $||\mathbf{x}|| < \delta$. Assume that $\exists\tilde{\mathbf{x}} \in \mathbb{R}^n_+$ such that $f(\tilde{\mathbf{x}}) \neq f(\mathbf{0}) = C$. Given $0 < \bar{\varepsilon} := | f(\tilde{\mathbf{x}}) - C |$, by the continuity of f at the origin $\exists\bar{\delta}$ such that $| f(\mathbf{x}) - C | < \bar{\varepsilon}\ \forall x \in \mathbb{R}^n_+$ with $||\mathbf{x}|| < \bar{\delta}$. Let t^* be small enough such that $||t^*\mathbf{x}|| < \bar{\delta}$. Hence $| f(\tilde{\mathbf{x}}) - C | = | f(t^*\tilde{\mathbf{x}}) - C | < \bar{\varepsilon} = | f(\tilde{\mathbf{x}}) - C |$, a contradiction. Thus, $f(\mathbf{x}) = C\ \forall \mathbf{x} \in \mathbb{R}^n_+$.[13]

[13] In Ferrarese (2021), when either the valuation or the impact function takes a positive value at the origin, the analysis is carried out for cases in which there exists a discontinuity at such point: a constant valuation would violate monotonicity.

We must show that $\bar{x}$ is the best reply of player i to $\bar{\mathbf{x}}_{-i}$. The first requirement is that any player i is strictly better off in $\bar{\mathbf{x}}$ rather than after deviating from $\bar{\mathbf{x}}$ by selecting $x_i = 0$ (the corner of the set of available strategies). Hence, we compare $x_i = \bar{x}$ and $x_i = 0$, given that $\mathbf{x}_{-i} = \bar{\mathbf{x}}_{-i}$. If $x_i = \bar{x}$, player i's payoff is:

$$\sum_{h\in\mathscr{H}} f_{i,h}(\bar{\mathbf{x}}) - \mathscr{C}_i(\bar{\mathbf{x}}). \tag{A-1}$$

By homogeneity, (A-1) becomes:

$$\sum_{h\in\mathscr{H}} \bar{x}^{\alpha_h} f_{i,h}(\mathbf{1}) - \bar{x}^s \mathscr{C}_i(\mathbf{1}).$$

If instead player i selects $x_i = 0$ his payoff is:

$$\sum_{h\in\mathscr{H}} f_{i,h}(0, \bar{\mathbf{x}}_{-i}) - \mathscr{C}_i(0, \bar{\mathbf{x}}_{-i}). \tag{A-2}$$

By homogeneity, (A-2) becomes:

$$\sum_{h\in\mathscr{H}} \bar{x}^{\alpha_h} f_{i,h}(0, \mathbf{1}_{-i}),$$

since $\mathscr{C}_i(0, \mathbf{1}_{-i}) = 0$. Thus, $\bar{x}$ is a better option than 0 if and only if:

$$\sum_{h\in\mathscr{H}} \bar{x}^{\alpha_h} f_{i,h}(\mathbf{1}) - \bar{x}^s \mathscr{C}_i(\mathbf{1}) > \sum_{h\in\mathscr{H}} \bar{x}^{\alpha_h} f_{i,h}(0, \mathbf{1}_{-i}). \tag{A-3}$$

Dividing both sides of (A-3) by $\bar{x}^s$, and according to Lemma 1, yields:

$$\frac{1}{\mathscr{C}_i(\mathbf{1})} \sum_{h\in\mathscr{H}_d\cup\mathscr{H}_{-d_0}} \left(\frac{f_{i,h}(\mathbf{1}) - f_{i,h}(0, \mathbf{1}_{-i})}{\bar{x}^{s-\alpha_h}} \right) > 1,$$

namely condition (I).

The second condition imposes quasiconcavity of the function $u_i\left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i}\right)$ and that the equilibrium candidate lies on its concave region. By homogeneity, the second derivative of this function with respect to x_i, given $\mathbf{x}_{-i} = \bar{\mathbf{x}}_{-i}$, is:

$$\left.\frac{\partial^2 u_i}{\partial x_i^2}(x_i, \mathbf{x}_{-i})\right|_{\mathbf{x}_{-i}=\bar{\mathbf{x}}_{-i}} = \sum_{h\in\mathscr{H}} \bar{x}^{\alpha_h - 2} \frac{\partial^2 f_{i,h}}{\partial x_i^2}\left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i}\right) - \bar{x}^{s-2} \frac{\partial^2 \mathscr{C}_i}{\partial x_i^2}\left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i}\right).$$

Using Lemma 1, condition (II) in the proposition is equivalent to $\frac{\partial^2 u_i}{\partial x_i^2}(x_i, \bar{\mathbf{x}}_{-i})$ being (possibly) initially positive and eventually negative. This involves quasi-concavity of $u_i(x_i, \bar{\mathbf{x}}_{-i})$.

Proof (Proposition 3) The null vector is not a SPNE if some player i has incentives to deviate by selecting an infinitesimal x_i. This deviation is profitable when there exists an $\hat{x} > 0$ such that $u_i(x_i, \mathbf{0}_{-i}) > u_i(\mathbf{0})$ for all $x_i \in (0, \hat{x})$. By homogeneity, this condition can be written as:

$$\sum_{h\in\mathscr{H}} x_i^{\alpha_h} f_{i,h}(1, \mathbf{0}_{-i}) - x_i^s \mathscr{C}_i(1, \mathbf{0}_{-i}) > \sum_{h\in\mathscr{H}} f_{i,h}(\mathbf{0}), \text{ for all } x_i \in (0, \hat{x}).$$

Distinguishing the different types of functions $f_{i,h}$, the previous condition can be rewritten as:

$$\begin{aligned} &\frac{1}{\mathscr{C}_i(1, \mathbf{0}_{-i})} \left(\sum_{h\in\mathscr{H}_{-d}} \left(\frac{f_{i,h}(1, \mathbf{0}_{-i})}{x_i^{s-\alpha_h}} - \frac{f_{i,h}(\mathbf{0})}{x_i^s} \right) + \right. \\ &\left. \sum_{h\in\mathscr{H}_d\cup\mathscr{H}_{-d_0}} \left(\frac{f_{i,h}(1, \mathbf{0}_{-i})}{x_i^{s-\alpha_h}} - \frac{f_{i,h}(\mathbf{0})}{x_i^s} \right) \right) > 1, \text{ for all } x_i \in (0, \hat{x}). \end{aligned} \tag{A-4}$$

Since any positive constant function is homogeneous of degree zero,

$$\sum_{h\in\mathscr{H}_{-d}} \left(\frac{f_{i,h}(1, \mathbf{0}_{-i})}{x_i^{s-\alpha_h}} - \frac{f_{i,h}(\mathbf{0})}{x_i^s} \right) = \sum_{h\in\mathscr{H}_{-d}} \left(\frac{f_{i,h}(1, \mathbf{0}_{-i}) - f_{i,h}(\mathbf{0})}{x_i^s} \right).$$

Notice also that Lemma 1 implies that $\sum_{h\in\mathscr{H}_{-d}} \left(\frac{f_{i,h}(1, \mathbf{0}_{-i}) - f_{i,h}(\mathbf{0})}{x_i^s} \right) = 0$. In consequence, (A-4) simplifies to:

$$\frac{1}{\mathscr{C}_i(1, \mathbf{0}_{-i})} \sum_{h\in\mathscr{H}_d\cup\mathscr{H}_{-d_0}} \left(\frac{f_{i,h}(1, \mathbf{0}_{-i})}{x_i^{s-\alpha_h}} - \frac{f_{i,h}(\mathbf{0})}{x_i^s} \right) > 1, \text{ for all } x_i \in (0, \hat{x}). \tag{A-5}$$

Notice that there will exist an $\hat{x}$ such that (A-5) is satisfied if and only if

$$\frac{1}{\mathscr{C}_i(1,\mathbf{0}_{-i})} \times \lim_{x_i \to 0^+} \sum_{h \in \mathscr{H}_d \cup \mathscr{H}_{-d_0}} \left(\frac{f_{i,h}(1,\mathbf{0}_{-i})}{x_i^{s-\alpha_h}} - \frac{f_{i,h}(\mathbf{0})}{x_i^s} \right) > 1.$$

This will hold, for example, when the fraction with the denominator with the smallest exponent goes to infinity as x_i approaches to 0. We now consider the following three cases:

$\alpha_{\underline{h}} < 0$. (A-5) is satisfied if and only if $f_{i,\underline{h}}(1,\mathbf{0}_{-i}) > 0$.
$\alpha_{\underline{h}} = 0$. (A-5) is satisfied if and only if $f_{i,\underline{h}}(1,\mathbf{0}_{-i}) - f_{i,\underline{h}}(\mathbf{0}) > 0$.
$\alpha_{\underline{h}} > 0$. Here, we need to analyze three subcases:

First, if $\alpha_{\underline{h}} > s$ then

$$\lim_{x_i \to 0^+} \sum_{h \in \mathscr{H}_d \cup \mathscr{H}_{-d_0}} \left(\frac{f_{i,h}(1,\mathbf{0}_{-i})}{x_i^{s-\alpha_h}} - \frac{f_{i,h}(\mathbf{0})}{x_i^s} \right) = \lim_{x_i \to 0^+} \sum_{h \in \mathscr{H}_d} -\frac{f_{i,h}(\mathbf{0})}{x_i^s} < 0,$$

as $f_{i,h}(\mathbf{0}) \geq 0$. Hence, (A-5) cannot hold.
Second, if $\alpha_{\underline{h}} = s$ then

$$\lim_{x_i \to 0^+} \frac{f_{i,h}(1,\mathbf{0}_{-i})}{x_i^{s-\alpha_h}} = \begin{cases} f_{i,\underline{h}}(1,\mathbf{0}_{-i}) & \text{, if } h = \underline{h} \\ 0 & \text{, otherwise.} \end{cases}$$

Thus, given that $f_{i,h}(\mathbf{0}) = 0$ for any $h \in \mathscr{H}_{-d_0}$ and $f_{i,h}(\mathbf{0}) \geq 0$, condition (A-5) holds if and only if

$$\mathscr{C}_i(1,\mathbf{0}_{-i})^{-1} f_{i,\underline{h}}(1,\mathbf{0}_{-i}) > 1,$$

and

$$f_{i,h}(\mathbf{0}) = 0, \ \forall h \in \mathscr{H}_d.$$

Third, if $\alpha_{\underline{h}} < s$ and $f_{i,h}(\mathbf{0}) > 0$ for some $h \in \mathscr{H}_d$, then condition (A-5) cannot hold. Instead, if $f_{i,h}(\mathbf{0}) = 0$ for all $h \in \mathscr{H}_d$, then condition (A-5) holds if and only if $f_{i,\underline{h}}(1,\mathbf{0}_{-i}) > 0$.