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# Sequential Stock Return Prediction Through Copulas

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# Sequential Stock Return Prediction Through Copulas

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**Abstract**

In this paper we perform density prediction for the equity returns in a non-linear manner by employing a copula-based approach. The use of asymmetric copulas enables us to model asymmetric predictive densities and non-linear dependencies between equity returns and some predictor variable. We consider static, hierarchical and dynamic dependence structures, together with lagged returns or dividend yield as predictor variables. In our proposed approach, the copula parameter and the marginals are estimated simultaneously by using Sequential Monte Carlo techniques. We apply proposed models to daily log returns of 20 assets traded at the NYSE. We show that in terms of predictive log Bayes Factors the realized volatility based models are preferred on average to the stochastic volatility based models. Moreover, asymmetric copula is preferred by more assets than the symmetric copula, advocating the use of non-linear models. Also, dividend yield is a better predictor variable than the lagged returns overall, but this result is reversed if we consider a volatile period only. Finally, hierarchical dependence parameter structure is preferred to dynamic or static approaches.

**Keywords:** Bayes Factor; Sequential Monte Carlo; Particle filters

**JEL classification:** C58, C11, C53

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# 1 Introduction

This paper re-examines the issue of equity return predictability in a novel non-linear context. Even though the efficient market hypothesis states that equity returns are unpredictable, multiple empirical studies have demonstrated that some returns are, in fact, predictable, see Lettau and Ludvigson (2001), for example. Nonetheless, the task is not trivial and requires careful consideration. Producing accurate predictive distributions of the returns has major implications for investors when making portfolio decisions or measuring tail risk, such as Value-at-Risk, for example. The vast majority of the literature consider regression-type models for the equity returns, imposing a linear dependence structure between the returns and the explanatory variable(s). There is plenty empirical evidence that equity return response to some predictor variable does not necessary have to be linear, see Nam (2003), Kahra et al. (2018), for example. One way to incorporate such non-linearities is via structural breaks, regime-switches, etc., but then the number of model parameters increases dramatically and the model quickly loses its parsimonious representation.

Apart from non-linear dependence structure, accounting for estimation uncertainty and volatility timing is essential for improving return forecasts, as shown by Johannes et al. (2014). In their widely cited paper the authors compare a number of alternative models for equity returns and show that significant gains in return prediction are obtained only when the investor is Bayesian and time-varying volatility is included in the model. Therefore, in this paper we extend the work of Johannes et al. (2014) and propose to model the excess equity returns using copula functions, offering a parsimonious model that allows for non-linear dependencies between the return and some predictor variable. We also consider an alternative time-varying volatility measure extracted from the intra-day equity price data. By using historical data of 20 assets we show that an ensemble of these extra features improves equity return distribution forecasts, that consequently can be used by investor in constructing portfolios or calculating tail risk.

From the technical viewpoint, in this work we build on models described in Chen and Fan

(2006) and Bouyé et al. (2002) and apply the proposed method to return prediction problem. The motivating example is a work of Johannes et al. (2014), where the authors consider a number of competing models and estimation approaches (Bayesian versus frequentist), however, all the models considered in their work are inherently linear. Therefore, in this work we extend the models of Johannes et al. (2014) in three directions. First, apart from the stochastic volatility models, we also study realized volatility models and their role in return prediction. Also, next to the dividend yield as a predictor variable we also use lagged returns. Finally, we consider not only regression-type linear dependence structures, but we also propose a copula-based model that allows for asymmetric dependencies. We allow these dependence structures to be static, dynamic and hierarchical.

Copulas have been applied in many fields in both social and natural sciences, especially in the context of financial time series, see Patton (2009), for example, for an extensive review. Even though the majority of the copula-related literature focuses on modeling contemporaneous dependence between multiple time series, copulas also permit to model the temporal dependence of a univariate time series, as noted in Chen and Fan (2006), among others. The use of copulas in modeling temporal dependence of univariate time series relates to Markov processes and have been described in Darsow et al. (1992), Joe (2015), for example. By considering various possible marginal distributions with different copula specifications one can capture often observed features of univariate financial time series, such as skewness and fat tails. Moreover, depending on the copula family, it is possible to model non-linear temporal dependencies, as opposed to the standard linear regression-type models. Chen and Fan (2006) study univariate semiparametric stationary Markov models, defined via a parametric copula and nonparametric marginal distributions. The authors are interested in estimating and forecasting transition distributions of a univariate time series, that are completely characterized by the marginal distribution and the copula dependence parameter. By having a full transition distribution it is straightforward to extract any conditional moment and/or conditional quantiles, analytically or via Monte Carlo approximation. Ibragimov (2009), Beare (2010), Ibragimov and Lentzas (2017) also explore the relationship between Markov processes and a copula function for univariate time series. In terms of applications, Abegaz and Naik-Nimbalkar (2008) propose a dynamic copula model for a first order Markov time-series with an ARMA-like specification and apply their model to simulated and real data. Sokolinskiy and van Dijk (2011) forecast realized volatility through a semi-parametric copula realized volatility model, but disregard the forecasting power of a deterministic time-varying copula parameter.

Important to note that in this work we do not pursue multivariate time series analysis, since it is out of the scope of the paper. However, the proposed framework could be extended to multivariate case by assuming some structure for joint modeling of the univariate processes, discussed in this paper. Neither we consider portfolio allocation exercise, especially given that Johannes et al. (2014) already showed how a superior univariate density prediction for each asset separately translates into superior out-of-sample portfolio performance. Finally, same as in Johannes et al. (2014), our investor is fully Bayesian and model estimation is carried out in a simultaneous manner via Sequential Monte Carlo techniques, allowing for fast inference and consistent model comparison via Bayes Factors.

The paper is organized as follows. Section 2 introduces the proposed copula-based model and describes the estimation procedure. Section 3 summarizes the set-up of our empirical study and presents empirical findings. Section 4 concludes and gives an outlook on further generalizations.

# 2 Methodology

The construction of flexible multivariate distributions using copulas has started with the seminal work of Sklar (1959). It allows to combine a copula function with marginal distributions, which not necessary have to be the same and can be specified separately. Since then, copulas have been widely used in modeling temporal dependence between financial time series, because they can capture non-linear dependence, as opposed to the correlation coefficient. For a formal introduction and details on copulas the reader is referred to the books of Nelsen (2006) and Joe (2015), among others.

Nelsen (2006) defines copulas in the following manner. Consider a collection of random variables $Y_1, \ldots, Y_d$ with corresponding distribution functions $F_i(y_i) = P[Y_i \leq y_i]$ for $i = 1 \ldots, d$ and a joint distribution function $H(y_1, \ldots, y_d) = P[Y_1 \leq y_1, \ldots, Y_d \leq y_d]$. Then, according to a theorem by Sklar (1959), there exists a copula $C$ such that

$$H(y_1, \ldots, y_d) = C(F_1(y_1), \ldots, F_d(y_d)). \tag{1}$$

In other words, it is possible to model univariate marginals and the dependence structure separately. Copulas are defined in the unit hypercube $[0, 1]^d$, where $d$ is the dimension of the data, and all univariate marginals are uniformly distributed $u_1, \ldots, u_d \overset{iid}{\sim} \mathcal{U}(0, 1)$, where $F_i(y_i) = u_i \forall i = 1, \ldots, d$. Copulas are very flexible in the sense that (i) the marginal distributions $F(\cdot)$ can be modeled independently from the dependence structure $C(\cdot)$ and (ii) copulas are able to capture asymmetric dependencies, as opposed to the standard multivariate distributions, such as Gaussian or Student's $t$. There is a vast selection of flexible bivariate one-parameter copulas, see Joe (2015) and Nelsen (2006) for example.

## 2.1 Copulas and Markov process for lagged returns as predictor variable

As described in Chen and Fan (2006), let $\{Y_t\}$ be a stationary first order Markov process whereas its probabilistic behavior is completely defined by joint distribution function $H(\cdot)$ between $Y_{t-1}$ and $Y_t$. On the other hand, as seen before, using Sklar's theorem, this joint can be expressed using a copula representation $H(y_t, y_{t-1}) = C(F(y_t), F(y_{t-1}); \theta)$, where $F(\cdot)$ is a marginal cumulative distribution function (CDF) of $Y_t$ and $\theta$ is a copula parameter. This allows to model a stationary Markov process using copula, where the transition kernel, determined by $\theta$, is constant over time.

Let $h(\cdot)$ be the joint density of $Y_t$ and $Y_{t-1}$, and $f(\cdot)$ the corresponding marginal probability density function (PDF) of $Y_t$. Using copula representation in (1), $h(\cdot)$ can be expressed as a product of the marginals and a copula density, which defines the dependence structure:

$$h(y_t, y_{t-1}) = c(F(y_t), F(y_{t-1}); \theta) \cdot f(y_t) \cdot f(y_{t-1}), \tag{2}$$

then, the conditional distribution of $y_t$ given $y_{t-1}$ is

$$f(y_t|y_{t-1}) = \frac{h(y_t, y_{t-1})}{f(y_{t-1})} = c(F(y_t), F(y_{t-1}); \theta) \cdot f(y_t). \tag{3}$$

Parameter $\theta$ completely determines the dependence structure which is constant across time. Then the collection of $\{Y_t\}$ follows a stationary first order Markov process with constant transition kernels. A natural extension is to relax the assumption of time-invariant dependence and consider dynamic copula approach by allowing $\theta$ to be time-varying, i.e. $\theta_t$. This implies that $\{Y_t\}$ is inhomogeneous first order Markov process with time-varying transition kernel. In this paper we consider static, hierarchical and dynamic models for the dependence coefficient $\theta_t$.

## 2.2 Copulas for dividend yield as predictor variable

Similarly to the case outlined above, for the predictor variable instead of lagged returns one can consider dividend yield, i.e. $y_{t-1}$ is replaced by $DY_{t-1}$ - the previous period's dividend yield. As seen in Lettau and Ludvigson (2001), Boudoukh et al. (2007), Johannes et al. (2014), dividend yield (or net payoff) is a reasonable predictor variable for the log returns. In this case the CDF of dividend yield $F_{DY}(DY_{t-1})$ has to be estimated separately. In this work we consider a non-parametric estimator of the distribution function of the dividend yield. The observed dividend yield data is transformed to the unit interval via empirical distribution function, adjusted by the $T/(T+1)$ factor, in order to avoid the unit at the end of the interval, as seen in Genest et al. (1995). Here $T$

is the length of the sample size.

$$u_t^{DY} = T^{-1}\sum_{j=1}^{T}\mathbf{1}\{DY_t \leq DY_j\}\frac{T}{T+1} = (T+1)^{-1}\sum_{j=1}^{T}\mathbf{1}\{DY_t \leq DY_j\},$$

where $\mathbf{1}\{\cdot\}$ is an indicator function. Then, $u_t^{DY} \overset{iid}{\sim} \mathcal{U}(0,1)$ for $t = 1, \dots, T$. Notice that there is no need to assume any dynamics for the dividend yield process, because at time $t$ the dividend yield from $t-1$ is observed. Next, we describe the assumed parametric model for the log returns.

## 2.3 Marginals for the returns

Define $r_t$ as the demeaned log-returns (in %) of some financial asset:

$$r_t = 100 \times \left(\log\frac{P_t}{P_{t-1}} - \mathrm{E}\left[\log\frac{P_t}{P_{t-1}}\right]\right), \tag{4}$$

where $P_{t-1}$ and $P_t$ are the prices at the beginning and at the end of the period, respectively. Also, define $RV_t = \sum_{j=1}^{N}\tilde{r}_{j,t}^2$ as a realized *ex post* volatility measure, where $\tilde{r}_{j,t}$ is a 10-minute intraday log-return for day $t$ and $N$ is the number of 10-minute intervals in a trading day, i.e. it holds that $r_t = \sum_{j=1}^{N}\tilde{r}_{j,t}$. In the empirical application section we have also tried 2, 5 and 15-minute returns for the $RV_t$ measure and from the signature plot we concluded that 10-minute intervals are big enough not to be affected by the market micro-structure noise and small enough to obtain efficient estimates. For introduction and review of realized volatility refer to Barndorff-Nielsen and Shephard (2002), Andersen et al. (2003), Barndorff-Nielsen and Shephard (2004), McAleer and Medeiros (2008), among others.

Then, the demeaned returns can be standardized via realized volatility measure and the resulting standardized returns are approximately Normally distributed, see Andersen et al. (2000), Andersen et al. (2001) (e.g. see Figure 1):

$$z_t = r_t/\sqrt{RV_t}, \quad \text{such that} \quad z_t \overset{approx}{\sim} \mathcal{N}(0,1). \tag{5}$$

Consequently, $\Phi(z_t) \equiv u_t^r \overset{iid}{\sim} \mathcal{U}(0,1)$. That is, the probability integral transforms of the returns, $u_t^r$, are uniformly distributed. Since we are interested not only in the model fit, but prediction as well, we need to specify a dynamic model for the $RV_t$. $\log(RV_t)$ can be modeled in many manners, such as simple AR(1) or a more sophisticated Heterogeneous Autoregressive model (HAR) of Corsi (2009) or HARQ of Bollerslev et al. (2016) specifications. We have tried fitting AR(1) and HAR models for several data sets and evaluating their predictive performance at $t+n$ via Bayes Factors. HAR model performs better for $t+n$ horizon, however, at $t+1$ we did not find any substantial improvement as compared with a simple AR(1) specification, since we re-estimated the model after each new data point. Therefore, the model used for the marginals of the returns is the following:

$$r_t = \epsilon_t^r \sqrt{\sigma_t^2}, \tag{6}$$

$$\log(\sigma_t^2) = \mu^{(l)} + \phi^{(l)} \log(\sigma_{t-1}^2) + \tau^{(l)} \epsilon_t^{(l)}, \tag{7}$$

where $\sigma_t^2 \equiv RV_t$, $l = RV$, $\epsilon_t^r$ and $\epsilon_t^{RV}$ are independent $\overset{iid}{\sim} \mathcal{N}(0,1)$. For the sake of comparison with benchmark models, we also consider a Stochastic Volatility (SV) process for the variance of the returns, first introduced by Taylor (1982). In this case, $\sigma^2$ is replaced with $SV_t$ and $l = SV$ in (6)-(7). SV-based models were considered in Johannes et al. (2014). Call $\Theta_V = (\mu^{(l)}, \phi^{(l)}, \tau^{(l)})$ a set of volatility-related parameters, where $l = \{SV, RV\}$. Finally, if the marginal distributions are specified correctly, the probability integral transform should provide uniformly distributed variables, see Diebold et al. (1998).

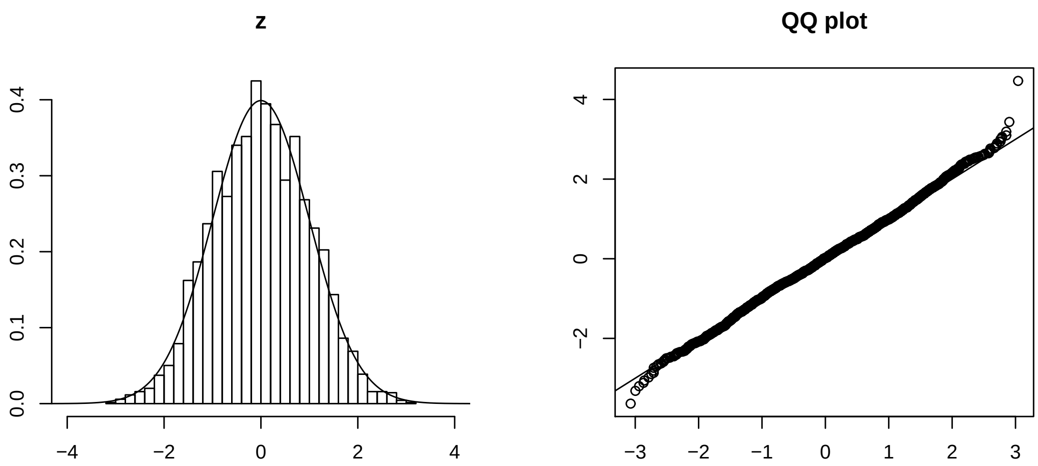


Figure 1: Demeaned returns of IBM, standardized by a RV mueasure.

### 2.4 Copula

In this paper we consider some of the most popular one-parameter copulas (only bivariate cases), such as Gaussian and Clayton (referred as Mardia-Takahasi-Clayton-Cook-Johnson copula in Joe, 2015). In the very first draft of the manuscript we have also considered a Gumbel copula. Gumber copula allows for strong upper tail dependence. Gumbel copula always performed the worst, therefore, we have decided to drop it in order to reduce the number of possible models. Since in this work we consider Markov process of order one, bivariate copulas with a single parameter are sufficiently flexible. On the other hand, if we wish to consider a higher-order dependence structure, one-parameter copulas might be too restrictive, because it would imply the same strength of dependence between all lags of returns. Gaussian copula is symmetric and does not present any tail dependence, meanwhile Clayton copula can model lower tail dependence. Economic interpretation of strong upper tail dependence is the following: when the markets are in turmoil, the dependence between log returns and some predictor variable becomes stronger as compared to the calm periods. The instability (non-constant regression coefficient) in return models that are based on dividend yield has been widely documented in the financial literature, see Goyal and Welch (2003), Paye and Timmermann (2006), Ang and Bekaert (2007), among others.

An important notion associated with copulas is Kendall's $\tau_\kappa$, a measure of dependence, which is given by $\tau_\kappa = 4\int\int_{I^2} C(u,v)dC(u,v) - 1$, see Nelsen (2006). See Table 1 for the CDFs, PDFs and Kendall's $\tau$s for Gaussian and Clayton copulas, and for detailed properties of these copulas refer to Joe (2015).

The copula parameter $\theta$ for different copula families lies in different domains. Therefore, in order to be able to compare the dependence across different copulas, we rely on Kendall's $\tau_\kappa$, where the is a one-to-one relationship between $\theta$ and $\tau_\kappa$. But first we need to make sure it lies in the same domain for all copulas of interest. Note, that for Gaussian copula $\tau_\kappa \in [-1,1]$, however, for standard Clayton copula $\tau_\kappa > 0$. Therefore, instead of considering standard Clayton copula ($c_C$), we couple this copula with its rotation and obtain the rotated Clayton copula ($c_{RC}$), that is defined as follows:

$$c_{RC}(u,v;\theta) = \begin{cases} c_C(u,v;\theta) \text{ if } \theta \geq 0, \\ c_C(1-u,v;-\theta) \text{ if } \theta < 0. \end{cases} \tag{8}$$

In order to be able to model the dynamics of $\tau_\kappa$ in an unconstrained manner, it is common to perform some deterministic transformations on this parameter and then model the behavior of this transformation $x$. In order to recover the copula parameter $\theta$ from the auxiliary latent variable $x$, we first convert $x$ to Kendall's $\tau_\kappa$ thought the inverse of Fisher's z-transformation $\tau_\kappa = \exp\{2x-1\}/\exp\{2x+1\}$ and then the resulting $\tau_\kappa$ is converted to a copula parameter $\theta$ through one-to-one copula-specific function (see Table 1). We consider two specifications for the copula parameter: hierarchical and dynamic.

**Hierarchical copula.** For the hierarchical parameter case, $\theta$, (its auxiliary process $x$) is fixed in time with a prior distribution $\mathcal{N}(m_x, V_x)$. In order to be completely uninformative we construct a hierarchical structure for $x$ by putting a Normal-Inverse Gamma $\mathcal{NIG}$ hyperprior on $m_x$ and

$V_x$. In this manner $\theta$ (or $x$) is treated as a latent variable that needs to be filtered out during the estimation procedure. Given the filtered $x$, the corresponding parameters $m_x$ and $V_x$ can then be estimated.

**Dynamic copula.** In dynamic case, the copula parameter $\theta_t$ (or its auxiliary process $x_t$) follows a random walk process: $x_t = x_{t-1} + V_x \eta_t, \quad \eta_t \overset{iid}{\sim} \mathcal{N}(0,1)$. The unknown parameter $V_x$ has an Inverse Gamma $\mathcal{IG}$ prior. In general, call $\Theta_C$ the set of parameters, associated with the evolution of $x_t$. $\Theta_C$ in hierarchical model contains parameters $m_x$ and $V_x$, meanwhile in the dynamic case contains only the variance parameter $V_x$. Then, $\Omega = (\Theta_V, \Theta_C)$ is a complete set of model parameters.

## 2.5 Conditional densities

The complete model for the univariate log returns $r_t$, written in a state-space representation, looks as follows:

$$\begin{aligned} u_t &= \Phi(r_t/\sigma_t), \\ (u_t, u_{t-1}^{(j)})|x_t &\sim C\left((u_t, u_{t-1}^{(j)}); \theta_t\right), \text{ where } \theta_t = f_\tau(x_t), \\ x_t|\Theta_C, x_{t-1} &\sim \mathcal{N}(x_t; m_x, V_x). \end{aligned} \tag{9}$$

Here $\sigma_t^2$ follows either $RV$ or $SV$ dynamics as in (6)-(7), $x_t$ is either hierarchical with fixed $m_x$ or follows a random walk process with $m_x = x_{t-1}$, $j \in \{DY, r\}$ and $f_\tau(\cdot)$ is a deterministic function that transforms the latent variable $x$ to the copula parameter $\theta$ via Kendall's $\tau$ transformation. Note that this model is specified for univariate log return series only, where the dependence between today's log return and some predictor variable, instead of relying on linear regression, is modeled via copula.

In many financial applications the estimation of parameters is not the ultimate goal. One is usually interested in estimating and forecasting conditional distributions and certain moments, such as the mean or variance for example. The predictive marginal density for one-step-ahead returns, given the all the information up till time $t$: $\mathcal{F}_{1:t} = (r_1, \ldots, r_t, DY_1, \ldots, DY_t, RV_1, \ldots, RV_t)$, where $j \in \{DY, r\}$, is:

$$f(r_{t+1}|\mathcal{F}_{1:t}) = \int\int c\left(\Phi\left(\frac{r_{t+1}}{\sqrt{\sigma^2_{t+1}}}\right), u_t^{(j)}\middle| x_{t+1}, \sigma^2_{t+1}\right) f(r_{t+1}|\sigma^2_{t+1}) f(x_{t+1}|\mathcal{F}_{1:t}) f(\sigma^2_{t+1}|\mathcal{F}_{1:t}) dx_{t+1} d\sigma^2_{t+1}. \tag{10}$$

Moreover, the conditional $k$th moment can be calculated as

$$\mathrm{E}[r^k_{t+1}|u_t^{(j)}] = \int r^k_{t+1} f(r_{t+1}|u_t^{(j)}) dr_{t+1}.$$

## 2.6 Estimation

The usual Bayesian estimation approach relies on MCMC schemes. Ausín and Lopes (2010), for example, use a multivariate random walk Metropolis - Hastings in a one-step estimation procedure for the parameters of the marginals and the copula, where their time-varying copula parameter is observation driven. Meanwhile, Almeida and Czado (2012) employ a two-step estimation approach where the marginal series are estimated first. Then, conditioning on the estimated marginal parameters, use a similar method as Ausín and Lopes (2010) to model copula dynamics, and a coarse grid method for updating the unobserved states of the stochastic copula. Creal and Tsay (2015) also employ a MCMC estimation scheme for modeling large panels of financial assets using high dimensional dynamic stochastic copula models. MCMC methods are inherently non-sequential and once a new data point is observed the algorithm has to be re-run all over again. Johannes et al. (2014), on the other hand, consider a sequential estimation approach for their proposed return prediction models. Therefore, in the spirit of their paper, we also employ an algorithm that performs sequential simultaneous estimation for the proposed model, where the marginals and the copula parameters are estimated simultaneously. In particular, we use a modified version of

Particle Learning of Carvalho et al. (2010a), that relies on the use of the sufficient statistics of Storvik (2002) to allow for parameter learning. The use of sufficient statistics has been shown to increase the efficiency of the algorithm by reducing the variance of sampling weights, see Carvalho et al. (2010a). For a detailed description of the algorithm see the Appendix A.1.

The priors for model parameters are chosen to be conditionally conjugate. For the set of copula parameters $\Theta_C$:

$$V_x \sim \mathcal{IG}(b_0/2, b_0 V_{x0}/2),$$
$$m_x | V_x \sim \mathcal{N}(m_m, V_m V_x).$$

And for the set of volatility parameters $\Theta_V$:

$$\tau^{2(l)} \sim \mathcal{IG}(b_0^{(l)}/2, b_0 \tau_0^{2(l)}/2),$$
$$\phi^{(l)} | \tau^{2(l)} \sim \mathcal{TN}_{(-1,1)}(m_\phi^{(l)}, V_\phi^{(l)} \tau^{2(i)}),$$
$$\mu^{(l)} \sim \mathcal{N}(m_\mu^{(l)}, V_\mu^{(l)}).$$

where $l = \{SV, RV\}$ and $\mathcal{TN}_{(a,b)}$ stands for a truncated Normal distribution with truncation points at $a$ and $b$. This restriction on the persistence parameter guarantees stationary process for the dependence parameter, however, it is not necessary (one could actually test if the process is stationary by removing the restriction). Initial states are $x_0 \sim \mathcal{N}(c_0^x, C_0^x)$ and $h_0 \sim \mathcal{N}(c_0^h, C_0^h)$ (only for SV model). Here $c_0^x$, $C_0^x$, $c_0^h$, $C_0^h$, $b_0$, $b_0 V_{x0}$, $m_m$, $V_m$, $b_0^{(l)}$, $b_0 \tau_0^{2(l)}$, $m_\phi$, $V_\phi$, $m_\mu$, $V_\mu$ are the known hyper-parameters. The initial states at $t_0$ for all parameters and latent variables are simulated from their corresponding priors. Then, the SMC algorithm transports the set of $N$ particles from time $t-1$ to time $t$, where at each step these particles are updated using the new information at time $t$. A set of particles is an approximate sample from the posterior distribution.

A well-known limitation of Particle filters is called particle degeneracy: an ever-decreasing set of unique atoms in the particle approximation of the density of interest. It has been shown in numerous studies that increasing the number of observations will lead to degenerating paths, unless the number of particles is being increased simultaneously. Therefore, particle degeneracy has to be monitored carefully and this shortcoming can be seen as a trade-off between the sequential nature of the algorithm and stability of MCMC for very large samples. For a general introduction to particle filters, comparison with MCMC, numerous empirical illustrations and discussion of the shortcomings of such estimation approach refer to Carvalho et al. (2010a), Carvalho et al. (2010b), Chopin et al. (2011), Lopes and Tsay (2011), Virbickaite et al. (2019), among many others.

Finally, if the interest is not on-line type inference, MCMC is still a gold standard. Recently other approaches, such as Particle MCMC or $SMC^2$ (that considers one SMC filter embedded into another - hence the name), have been emerging, presenting an alternative to the proposed estimation scheme, see Andrieu et al. (2010), Pitt et al. (2012), Chopin et al. (2013), Fulop and Li (2013), among others.

## 2.7 Evaluation

The model comparison is carried out via sequential predictive log Bayes Factors (BF). As pointed out in Koop (2003), Bayes Factors permit consistent model comparison even for non-nested models. Also, it contains rewards for model fit, accounts for coherency between the prior and the information arising from the data, as well as rewards parsimony. Bayes Factor between two competing models $\mathcal{M}_k, \quad k \in \{1, 2\}$, as seen in Kass and Raftery (1995), is defined as

$$BF_{12} = \frac{p(D|\mathcal{M}_1)}{p(D|\mathcal{M}_2)},$$

where $p(D|\mathcal{M}_k)$ is the marginal likelihood for data $D$ given a model $\mathcal{M}_k$. Then the log predictive Bayes Factor at time $t$ is defined as

$$\log BF_{12,t} = \sum_{s=1}^{t} \log p(r_s|\mathcal{M}_1) - \sum_{s=1}^{t} \log p(r_s|\mathcal{M}_2),$$

and the marginal predictive $p(r_t|\mathcal{M}_k)$ for model $k$ is obtained as in (10). The integral in (10) is rarely analytically tractable and can be approximated using costly MCMC-based procedures. On the other hand, sequential Monte Carlo approach produces marginal likelihoods as by-products of the estimation procedure, therefore, sequential model comparison via log predictive Bayes Factors can be carried out without any additional computational cost.

Alternatively, we also employ a log predictive score ($LPS$) measure, which is defined as follows:

$$LPS_T^{(i)} = T^{-1} \sum_{t=1}^{T} \log p(r_t|\mathcal{M}_k, \Omega_k),$$

which can be seen an average log predictive Bayes Factor at the end of the sample period. Notice, that $\Omega_k$ is estimated using the data available up till time $t-1$, therefore, $LPS$ is a predictive, not an in-sample score. One could also consider other comparison metrics, such as value-at-risk measure for example. However, (marginal) predictive likelihoods are more informative in the sense that they focus on the entire distribution of the returns, not only on the tails.

# 3 Empirical Analysis

## 3.1 Data and Set-up

We investigate the forecast performance of the proposed copula model for 20 daily log returns of US stocks traded at the NYSE, that are coded as (the full names are available in Table 2):

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| "AIG" | "BA" | "BAC" | "C" | "F" | "GE" | "GS" | "HD" | "HPQ" | "IBM" |
| "JNJ" | "JPM" | "KO" | "MO" | "NKE" | "PFE" | "PG" | "VZ" | "WMT" | "XOM" |

We consider daily observations from 01/01/2001 - 31/12/2015 and at each point in time perform one-step-ahead marginal density prediction. We also consider two sub-periods in order to get a better understanding how the models perform during calm (Jan 2002- Dec 2006) and nervous (Jan 2007 - Dec 2012) periods. The competing models are all possible combinations of the following, resulting into 16 models:

1. Realized volatility (RV) or stochastic volatility (SV) based models.
2. Normal Copula (NC) or Clayton Copula (CC) models.
3. Lagged returns (r) or dividend yield (DY) as a predictor variable.
4. Hierarchical (h) and dynamic (d) dependence structures.

Note that Normal copula based model allows for linear (regression-type) dependence structure. Finally, for comparison purposes we include four static linear models oftenly used as benchmarks, similar to the ones seen in Johannes et al. (2014):

$$r_t = \theta r_{t-1} + \epsilon_t^r \sigma_t^{(l)} \tag{11}$$

$$r_t = \theta DY_{t-1} + \epsilon_t^r \sigma_t^{(l)} \tag{12}$$

The four models are all static (s) in $\theta$, RV or SV based such that $l = \{SV, RV\}$, using either returns (11) or dividend yield (12) as a predictor variable. These static models are also estimated using the SMC scheme. The dependence parameter $\theta$ between the returns and the predictor variable is assumed to have a normal prior distribution with some known hyper-parameter values. Our proposed hierarchical structure is more flexible in the sense that the hyper-parameters have their own priors and are estimated, also, $\theta$ is a latent variable that is filtered out. We show in the real data application that for all assets the proposed hierarchical structure dominates the static and dynamic models. For a complete list of models refer to Table 3. The hyper-parameter values are as follows: $c_0^x = 0$, $C_0^x = 0.1$, $c_0^h = RV_1$, $C_0^h = 0.36$, $b_0 = 6$, $b_0 V_{x0} = 0.004$, $m_m = 0$, $V_m = 0.1$, $b_0^{(l)} = 3$, $b_0 \tau_0^{2(l)} = 0.36$, $m_\phi = 0.85$, $V_\phi = 1$, $m_\mu = 0$, $V_\mu = 0.01$. The values were chosen either to represent uninformative priors, or to match the unconditional sample moments (for example, the variance of $RV$), or by employing previous knowledge from numerous empirical studies (for example, it is known that persistence parameter in $RV$ or $SV$ models is close to 1). The number of particles is set to be very large, $N = 500$k, in order to ensure that there are no particle degeneracy problems. Model monitoring can be performed via sequential log predictive Bayes Factors.

## 3.2 Results

Since there are 20 models and 20 assets, there are $20 \times 20$ estimation results (only for $LPS$) to be reported. The corresponding tables can be found in the Appendix A.2. These tables include the complete model list and report the $LPS$ for all 20 models and 20 assets. Tables 3, 4 and 5 report the $LPS$ for all models and all assets for the entire sample and two subsamples: calm (Jan 2002-Dec 2006) and volatile (Jan 2007 - Dec 2012) periods. The numbers in bold indicate the highest LPS for each asset. For now we will not discuss the magnitude of these differences (this is done in the later paragraphs), but rather see how many times one or another model is preferred.

Firstly, return based models and dividend yield based models appear as best models for equal number of assets (10$\div$10, such that $r \div DY$) during the entire period. An interesting finding though is that for the calm period, see Table 4, dividend yield based models are the best for 16 out of 20 assets (4$\div$ 16). However, for the crisis period, see Table 5, this ratio almost reverses: now return based models are better for a bigger number of assets as compared to the dividend yield based models ($13 \div 7$). This has some important implications: even though for the entire sample dividend yield is a better predictor variable for the majority of the assets, this result might not hold depending if we are considering calm or crisis period. For calm period dividend yield seems to provide better predictive power, meanwhile during crisis periods lagged return is a better predictor variable for the majority of the assets.

As for the dependence structure, hierarchical model (h), as compared to dynamic (d) or static (s) models, is always preferred: in full sample and both subsamples. In the full sample and during the crisis period, hierarchical models are the best for 18 and 17 assets (out of 20), respectively. Only for the calm period hierarchical model is the best for 10 assets, meanwhile dynamic and static models share the rest, 3 and 7 assets respectively. This result implies that for calm period it is reasonable to consider less flexible dependence structures, however, for the crisis period a more flexible hierarchical dependence structure is preferred, as expected.

Next, we present four main figures, that can summarize the principal findings of the paper for the entire period taking into account the magnitude of the preferences via sequential predictive log Bayes Factors. In the following figures we report the number of assets that prefer one or another model type. Preference is measured in terms of differences between as expected , and if $LPS_{A,i} - LPS_{B,i} > 0$, it is said that model $A$ is preferred to model $B$ by the asset $i$. This classification loses information about the strength of the preference. Therefore, we also report the *average* sequential predictive log Bayes Factor. Averaging across assets is not intuitively appealing, however, it can convey important information on the average strength of the preference.

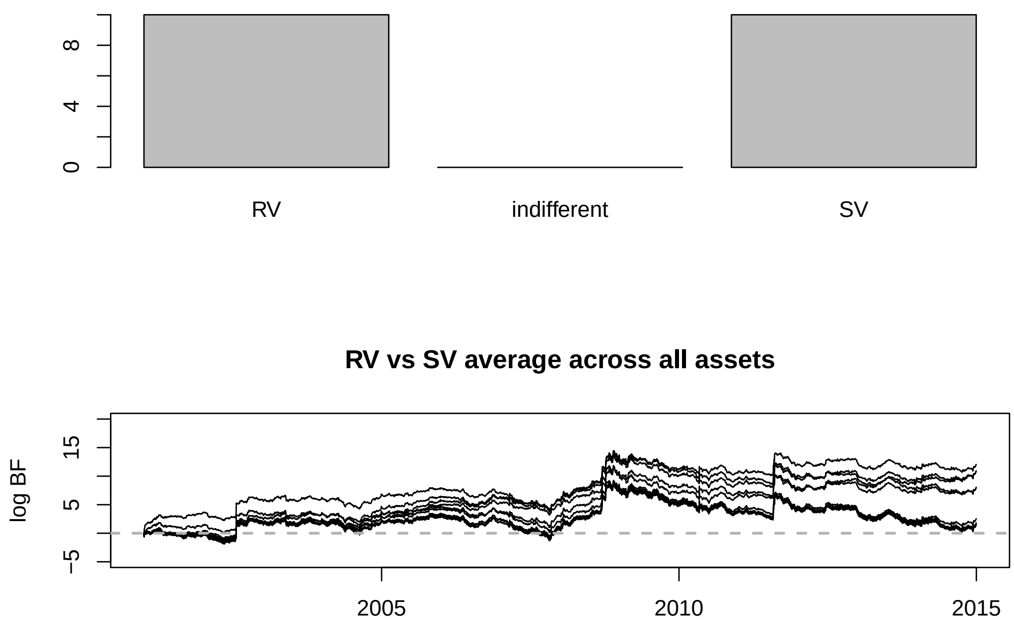


Figure 2: RV vs. SV-based models.

Figure 2 compares how many assets in total prefer RV versus SV-based models. Out of 20 models in total, half of them (ten) are RV-based, and the other half are SV-based (for a complete list of models refer to Table 3 in the Appendix A.2). For example, model (1) is compared to its 'counterpart' model (5), which both are the same in almost every aspect, except for volatility modeling approach, i.e. RV vs. SV. Then, for asset $i$, we perform ten comparisons and if RV-based model is preferred $> 5$ times, we say that asset $i$ prefers RV-based models. If asset $i$ prefers RV-based models half of the time, then we say that asset $i$ is indifferent. As seen in the top part of the Figure 2, exactly for half of the assets RV-based models perform better, and the other half prefer the SV-based models. The bottom plot of the figure draws sequential predictive log Bayes Factor, averaged across all assets, for 10 RV vs. SV-based model combinations. Here we can see that even though half of the assets prefer one or another volatility model, the preference for RV-based models is much stronger on average. A log Bayes Factor $> 5$ indicates a strong preference from one model over another, see Kass and Raftery (1995) for the interpretation of the Bayes Factors. This is an important result in the sense that availability of high frequency data can improve predictive performance of the one step ahead log returns. Finally, one can see an increase in sequential predictive log Bayes Factors around the year 2008, meaning that the predictive power of RV-based models increases during financial turmoils.

- Assets that prefer the RV-based models are: "AIG" "BA" "BAC" "C" "F" "GS" "HPQ" "JPM" "NKE" "XOM".

- Assets that prefer the SV-based models are: "GE" "HD" "IBM" "JNJ" "KO" "MO" "PFE" "PG" "VZ" "WMT".

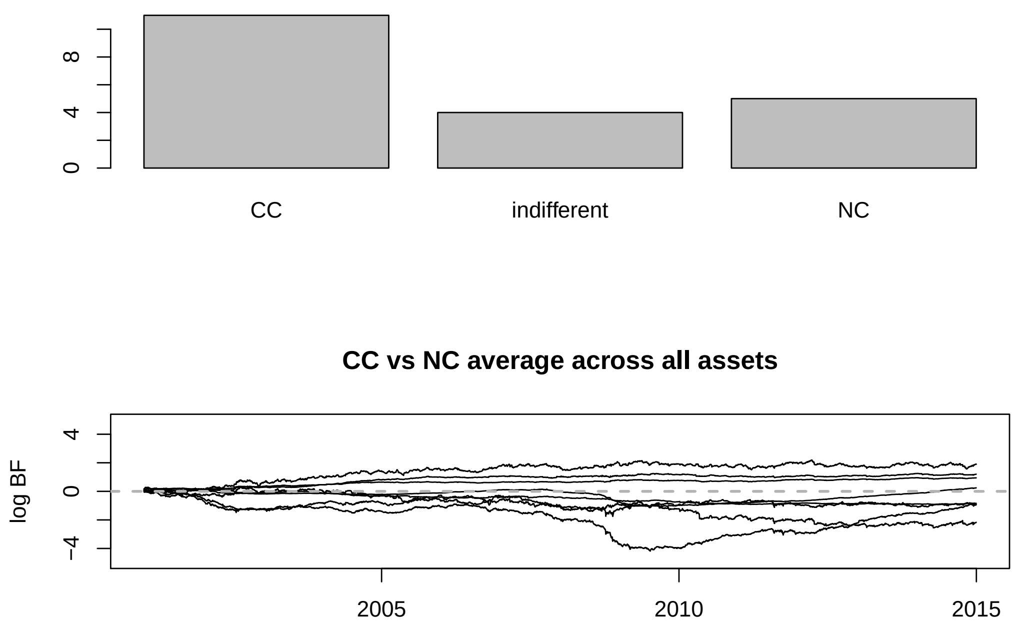


Figure 3: Clayton copula (CC) vs. Normal copula (NC)-based models.

Figure 3 compares how many assets prefer Clayton copula (CC) based models vs. Normal copula (NC) based models. Out of 20 models in total, 8 are CC-based and 8 are NC-based. In the top plot we can observe that more assets prefer CC-based models. Also, there are 4 assets in the 'indifferent' category, meaning that half of the time these assets preferred CC, another half NC-based models. The bottom plot of the figure indicates that the preference on average is not conclusive, meaning that it is asset-specific. In other words, some assets exhibit symmetric and others - asymmetric dependence structures. As mentioned before, the first draft of the manuscript also included Gumbel copula based models, however, it always performed the worst and due to space restrictions we are not reporting estimation results.

- Assets that prefer the CC-based models are: "AIG" "BA" "F" "GS" "HD" "JNJ" "KO" "PFE" "PG" "WMT" "XOM".

- Assets that are indifferent to CC vs. NC-based models are: "BAC" "HPQ" "IBM" "NKE".

- Assets that prefer the NC-based models are: "C" "GE" "JPM" "MO" "VZ".

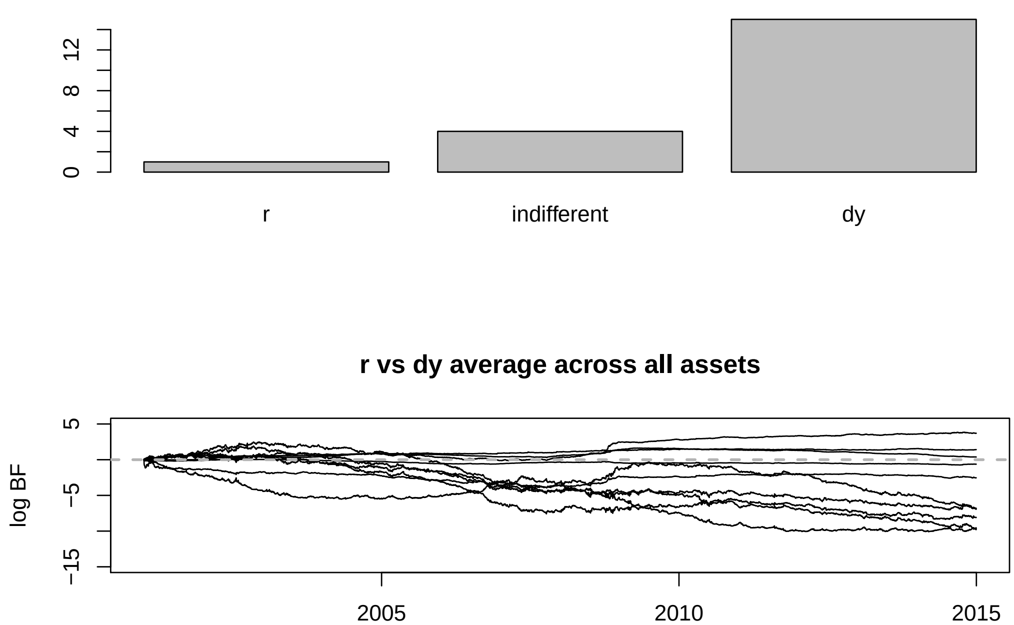


Figure 4: Lagged returns (r) vs. dividend yield (DY)-based models.

Next, Figure 4 presents which is a more powerful predictor for the daily log returns: lagged log returns or lagged dividend yield. In total there are 10 lagged return-based and 10 dividend yield-based models. As seen from the top plot, for the majority of the assets dividend yield is a stronger predictor variable than the lagged returns. The strength of the preference is moderate, in the sense that bottom plot sequential predictive log Bayes Factors seem to favorite dividend yield based models.

- Assets that prefer the lagged returns-based models are: "BA".

- Assets that are indifferent to lagged returns vs. dividend yield as predictor variable are: "AIG" "HPQ" "JNJ" "VZ".

- Assets that prefer the dividend yield-based models are: "BAC" "C" "F" "GE" "GS" "HD" "IBM" "JPM" "KO" "MO" "NKE" "PFE" "PG" "WMT" "XOM".

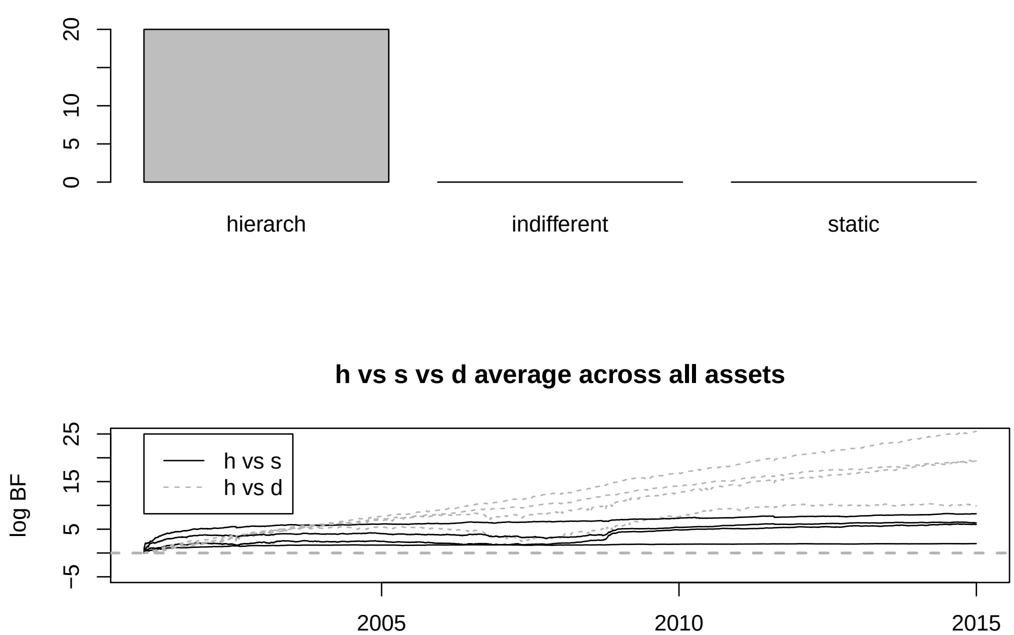


Figure 5: hierarchical (h) vs. static (s) vs. dynamic (d) models.

Figure 5 compares only Normal copula based (i.e. linear dependence structure) models across different ways of modeling the dependence parameter: hierarchical (h) and dynamic (d). Also, we include the four linear static (s) models, that are defined in (11) and (12). Note that in a static setting a simple linear model is used (similarly as seen in Johannes et al., 2014) there is a single parameter to be estimated. Hierarchical dependence structure allows more flexibility, since the hyper-parameters (mean and variance) are estimated and the parameter, governing dependence structure is treated as a latent random variable. Finally, dynamic model follows a random walk structure, where apart from filtering the latent variable we also estimate the variance parameter. As seen from the bottom plot of the Figure 5, the hierarchical model (h) on average is strongly preferred to the dynamic model (d) and preferred to the static (s) dependence structure. The top plot of the figure indicates that all assets prefer hierarchical rather than static dependence structures. In general, this figure summarizes two important results. Firstly, contrary to Johannes et al. (2014), dynamic dependence structure produces lower Bayes Factors than the static dependence structure. Although important to notice that Johannes et al. (2014) investigate monthly data, whereas our results are based on daily data. Secondly, hierarchical dependence structure seems to be the most flexible, at least for daily data.

# 4 Concluding Remarks

In this paper, we consider hierarchical and time-varying stochastic copulas to model dependence between a single financial asset and two alternative predictor variables: its lagged value and dividend yield. We have designed a fast one-step estimation procedure based on the SMC techniques, that allow for consistent model comparison via log predictive Bayes Factors. We apply the proposed models to daily log returns of 20 assets traded at the NYSE and we find a number of important results.

Firstly, on average, RV-based models outperform the SV-based models in terms of sequential predictive log Bayes Factors. Moreover, more assets exhibit asymmetric dependence structure preferring Clayton copula to Normal copula models. Also, for majority of the assets dividend yield is a better predictor variable than the lagged returns. Finally, flexible hierarchical dependence structure is preferred by all assets versus dynamic random walk or static dependence structures.

Considering two sub-periods separately (calm and volatile), we have found that for calm period dividend yield is a better predictor variable, meanwhile during the crisis periods lagged return is a better predictor variable for the majority of the assets. As for the dependence structure, we find that for calm period it is reasonable to consider less flexible dynamics for the dependence parameter, however, for the crisis period a more flexible hierarchical dependence structure is preferred.

## Outlook

The possible extensions is to specify a HAR model for forecasting RV, and the distribution of the returns, for longer horizons. This should be done together with departing from bivariate copula to multivariate case that would permit to consider more than one lag of the log returns as well as the dividend yield simultaneously. Another line of extensions could consider multivariate extensions and perform portfolio allocation exercise.

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# A Appendix

## A.1 Technical Appendix

For a state-space model described in (9) particle filter solves hidden state filtering problem for a set of fixed parameters $\Theta_C$ and $\Theta_V$. A set of sufficient statistics $S_t$ contains all updated hyper-parameters, necessary for the parameter simulation, as well as filtered state variables $x_t$, all approximated by a set of $N$ particles. For $t = 1 \ldots, T$ iterate through the following steps:

1. **(Blind) Propagating.**

   Sample new hidden states $x_{t+1}$ from $x_{t+1} \sim p(x_{t+1}|x_t, \Theta_C)$ and obtain $\theta_{t+1}$ deterministically. Sample $h_{t+1}$ from $p(h_{t+1}|h_t, \Theta_V)$ (only for the SV model).

   (a.1) For the hierarchical model, where $x = x_{t+1}$, the dependence on the hyperparamaters $m_x$ and $V_x$ can be integrated out analytically:

$$\begin{aligned}
p(x_{t+1}) &= \int\int p(x_{t+1}|m_x, V_x)p(m_x, V_x)dm_x dV_x \\
&= \int\int f_N(x_{t+1}|m_x, V_x)f_N(m_x|V_x)f_{IG}(V_x)dm_x dV_x \\
&= \frac{\Gamma\left(\frac{b_0+1}{2}\right)}{\Gamma\left(\frac{b_0}{2}\right)} (\pi(b_0 V_{x0})(V_m+1))^{-1/2} \left(1 + \frac{(x_{t+1} - m_m)^2}{(b_0 V_{x0})(V_m+1)}\right)^{-\frac{b_0+1}{2}},
\end{aligned}$$

   which implies that $x_{t+1}$ has a Student-$t$ distribution with $b_0 > 0$ degrees of freedom, location parameter $m_m$, and scale parameter $(b_0 V_{x0})(V_m + 1)/b_0 > 0$.

   (a.2) For the dynamic model for $x_{t+1}$ the dependence on the hyperparamater $V_x$, same as before, can be integrated out analytically:

$$
\begin{aligned}
p(x_{t+1}|x_t) &= \int p(x_{t+1}|x_t, V_x)p(V_x|x_t)dV_x \\
&= \int f_N(x_{t+1}|x_t, V_x) f_{IG}(V_x)dV_x \\
&= \frac{\Gamma\left(\frac{b_0+1}{2}\right)}{\Gamma\left(\frac{b_0}{2}\right)} (\pi(b_0 V_{x0}))^{-1/2} \left(1 + \frac{(x_{t+1}-x_t)^2}{(b_0 V_{x0})}\right)^{-\frac{b_0+1}{2}},
\end{aligned}
$$

which implies that $x_{t+1}$ has a Student-$t$ distribution with $b_0 > 0$ degrees of freedom, location parameter $x_t$, and scale parameter $(b_0 V_{x0})/b_0 > 0$.

2. **Resampling.**

   Resample old particles (parameters and the set of sufficient statistics, including states) with weights $w \propto p(u_{t+1}, u_t; \theta_{t+1})$, that are proportional to the predictive density of the $(u_{t+1}, u_t)$, where $u_{t+1} = \Phi(r_{t+1}/\sigma_{t+1}^{(l)})$, such that $l = \{SV, RV\}$. $u_t^{(j)}$ is either the transformed lagged return, or transformed dividend yield, i.e. $j \in \{r, DY\}$. The components of $\Theta_C$ and $\Theta_V$ have been simulated at the end of the previous period. The resampled particles are denoted by a tilde above the particle, as in $\tilde{\Theta}$, for example. Note, that $\sigma_{t+1}^{(l)}$ is either propagated in the previous step (for the SV model), or is observed (for the RV model).

3. **Propagating sufficient statistics and learning $\Theta_C$.**

   For the hierarchical model:

   (c.1) Sample $V_x$ from $\mathcal{IG}(V_x; b_0^\star/2, b_0^\star V_{x0}^\star/2)$, where

$$
b_0^\star = \tilde{b}_0 + 1 \quad \text{and} \quad b_0^\star V_{x0}^\star = \tilde{b}_0 \tilde{V}_{x0} + \frac{(\tilde{x}_{t+1} - \tilde{m}_m)^2}{1 + \tilde{V}_m}.
$$

   (c.2) Sample $m_x$ from $\mathcal{N}(m_x; m_m^\star, V_m^\star V_x)$, where

$$
m_m^\star = \frac{\tilde{m}_m + \tilde{V}_m \tilde{x}_{t+1}}{1 + \tilde{V}_m} \quad \text{and} \quad V_m^\star = \frac{\tilde{V}_m}{1 + \tilde{V}_m}.
$$

For the dynamic model:

(c.3) Sample $V_x$ from $\mathcal{IG}(V_x; b_0^\star/2, b_0^\star V_{x0}^\star/2)$, where

$$b_0^\star = \tilde{b}_0 + 1 \quad \text{and} \quad b_0^\star V_{x0}^\star = \tilde{b}_0 \tilde{V}_{x0} + (\tilde{x}_{t+1} - \tilde{x}_t)^2.$$

4. **Propagating sufficient statistics and learning** $\Theta_V$, where $h_{t+1}$ is either observed (RV) or filtered (SV) volatility, depending on the model.

(d.1) Sample $\tau^{2(i)}$ from $\mathcal{IG}(\tau^{2(i)}; b_0^{(i)\star}/2, b_0^\star \tau_0^{2\star}/2)$, where

$$b_0^\star = \tilde{b}_0 + 1 \quad \text{and} \quad b_0^\star \tau_0^{2\star} = \tilde{b}_0 \tilde{\tau}_0^2 + \frac{(\tilde{m}_\phi \tilde{h}_t - (\tilde{h}_{t+1} - \tilde{\mu}))^2}{1 + \tilde{V}_\phi \tilde{h}_t^2}.$$

(d.2) Sample $\phi$ from $\mathcal{N}(\phi; m_\phi^\star, V_\phi^\star \tau^2)$, where

$$m_\phi^\star = \frac{\tilde{m}_\phi + \tilde{V}_\phi \tilde{h}_t (\tilde{h}_{t+1} - \tilde{\mu})}{1 + \tilde{V}_\phi \tilde{h}_t^2} \quad \text{and} \quad V_\phi^\star = \frac{\tilde{V}_\phi}{1 + \tilde{V}_\phi \tilde{h}_t^2}.$$

(d.3) Sample $\mu$ from $\mathcal{N}(\mu; m_\mu^\star, V_\mu^\star)$, where

$$m_\mu^\star = \frac{\tilde{m}_\mu \tau^2 + \tilde{V}_\mu (\tilde{h}_{t+1} - \phi \tilde{h}_t)}{\tau^2 + \tilde{V}_\mu} \quad \text{and} \quad V_\mu^\star = \frac{\tau^2 \tilde{V}_\mu}{\tau^2 + \tilde{V}_\mu}.$$

## A.2 Tables

Table 1: Gaussian and Clayton copulas: CDF, PDF and Kendall's $\tau$s.

| | |
|---|---|
| | Gaussian |
| CDF | $C(u,v;\theta)=\Phi_2(\Phi^{-1}(u),\Phi^{-1}(v);\theta),\ \ \theta\in[0,1]$ |
| PDF | $c(u,v;\theta)=(1-\theta^2)^{-1/2}\exp\left\{\frac{2\theta xy-\theta^2(x^2+y^2)}{2(1-\theta^2)}\right\},\ x=\Phi^{-1}(u),\ y=\Phi^{-1}(v)$ |
| $\tau$ | $\tau_\kappa=2\arcsin(\theta)/\pi,\ \ \theta=\sin(\pi\tau_\kappa/2)$ |
| | Clayton |
| CDF | $C(u,v;\theta)=\Phi_2(\Phi^{-1}(u),\Phi^{-1}(v);\theta),\ \ \theta\in[0,1]$ |
| PDF | $c(u,v;\theta)=(1-\theta^2)^{-1/2}\exp\left\{\frac{2\theta xy-\theta^2(x^2+y^2)}{2(1-\theta^2)}\right\},\ x=\Phi^{-1}(u),\ y=\Phi^{-1}(v)$ |
| $\tau$ | $\tau_\kappa=2\arcsin(\theta)/\pi,\ \ \theta=\sin(\pi\tau_\kappa/2)$ |

Table 2: Abbreviations and the full names of the US stocks.

| | |
|---|---|
| "AIG" | American International Group, Inc. |
| "BA" | The Boeing Company |
| "BAC" | Bank of America Corporation |
| "C" | Citigroup Inc. |
| "F" | Ford Motor Company |
| "GE" | General Electric Company |
| "GS" | The Goldman Sachs Group, Inc. |
| "HD" | The Home Depot, Inc. |
| "HPQ" | HP Inc. |
| "IBM" | International Business Machines Corporation |
| "JNJ" | Johnson & Johnson |
| "JPM" | JPMorgan Chase & Co. |
| "KO" | The Coca-Cola Company |
| "MO" | Altria Group, Inc. |
| "NKE" | NIKE, Inc. |
| "PFE" | Pfizer Inc. |
| "PG" | The Procter & Gamble Company |
| "VZ" | Verizon Communications Inc. |
| "WMT" | Wal-Mart Stores, Inc. |
| "XOM" | Exxon Mobil Corporation |

Table 3: Average log predictive scores for 20 assets for 20 models (sample: 03.01.2001 - 01.12.2014)

| | AIG | BA | BAC | C | F | GE | GS | HD | HPQ | IBM |
|---|---|---|---|---|---|---|---|---|---|---|
| (1) NC-RV-d-r | -1.761 | -1.546 | -1.619 | -1.693 | -1.837 | -1.459 | -1.679 | -1.555 | -1.699 | -1.285 |
| (2) NC-RV-d-dy | -1.754 | -1.547 | -1.617 | -1.686 | -1.834 | -1.455 | -1.678 | -1.550 | -1.697 | -1.282 |
| (3) NC-RV-h-r | -1.756 | -1.542 | -1.613 | -1.687 | -1.831 | -1.453 | -1.673 | -1.549 | -1.692 | -1.278 |
| (4) NC-RV-h-dy | -1.757 | -1.542 | -1.613 | -1.686 | -1.831 | -1.452 | -1.673 | -1.548 | -1.692 | -1.278 |
| (5) NC-SV-d-r | -1.772 | -1.551 | -1.622 | -1.705 | -1.844 | -1.451 | -1.684 | -1.556 | -1.709 | -1.284 |
| (6) NC-SV-d-dy | -1.763 | -1.552 | -1.621 | -1.698 | -1.840 | -1.452 | -1.685 | -1.551 | -1.709 | -1.282 |
| (7) NC-SV-h-r | -1.766 | -1.546 | -1.616 | -1.698 | -1.836 | -1.445 | -1.678 | -1.546 | -1.703 | -1.277 |
| (8) NC-SV-h-dy | -1.766 | -1.546 | -1.615 | -1.696 | -1.835 | -1.445 | -1.677 | -1.546 | -1.702 | -1.277 |
| (9) CC-RV-d-r | -1.759 | -1.546 | -1.619 | -1.692 | -1.839 | -1.459 | -1.678 | -1.554 | -1.698 | -1.284 |
| (10) CC-RV-d-dy | **-1.754** | -1.548 | -1.618 | -1.688 | -1.834 | -1.456 | -1.677 | -1.550 | -1.700 | -1.283 |
| (11) CC-RV-h-r | -1.756 | -1.542 | **-1.613** | **-1.686** | -1.831 | -1.453 | -1.674 | -1.548 | **-1.692** | -1.278 |
| (12) CC-RV-h-dy | -1.756 | **-1.542** | -1.613 | -1.687 | **-1.831** | -1.451 | **-1.673** | -1.548 | -1.693 | -1.278 |
| (13) CC-SV-d-r | -1.772 | -1.552 | -1.624 | -1.706 | -1.844 | -1.456 | -1.684 | -1.553 | -1.709 | -1.287 |
| (14) CC-SV-d-dy | -1.760 | -1.554 | -1.622 | -1.701 | -1.838 | -1.452 | -1.684 | -1.550 | -1.709 | -1.283 |
| (15) CC-SV-h-r | -1.765 | -1.545 | -1.615 | -1.695 | -1.836 | -1.446 | -1.676 | **-1.546** | -1.702 | **-1.277** |
| (16) CC-SV-h-dy | -1.765 | -1.546 | -1.616 | -1.698 | -1.835 | **-1.444** | -1.676 | -1.546 | -1.703 | -1.277 |
| (17) L-RV-s-r | -1.756 | -1.543 | -1.613 | -1.687 | -1.832 | -1.453 | -1.674 | -1.549 | -1.693 | -1.279 |
| (18) L-RV-s-dy | -1.764 | -1.544 | -1.614 | -1.687 | -1.832 | -1.453 | -1.674 | -1.549 | -1.694 | -1.280 |
| (19) L-SV-s-r | -1.767 | -1.547 | -1.620 | -1.698 | -1.836 | -1.445 | -1.683 | -1.556 | -1.704 | -1.279 |
| (20) L-SV-s-dy | -1.767 | -1.548 | -1.617 | -1.697 | -1.835 | -1.449 | -1.678 | -1.547 | -1.704 | -1.278 |

| | JNJ | JPM | KO | MO | NKE | PFE | PG | VZ | WMT | XOM |
|---|---|---|---|---|---|---|---|---|---|---|
| (1) NC-RV-d-r | -1.016 | -1.643 | -1.138 | -1.266 | -1.489 | -1.390 | -1.070 | -1.362 | -1.224 | -1.316 |
| (2) NC-RV-d-dy | -1.011 | -1.641 | -1.133 | -1.263 | -1.487 | -1.387 | -1.069 | -1.361 | -1.223 | -1.315 |
| (3) NC-RV-h-r | -1.011 | -1.637 | -1.132 | -1.262 | -1.484 | -1.385 | -1.065 | -1.357 | -1.219 | -1.311 |
| (4) NC-RV-h-dy | -1.011 | -1.637 | -1.131 | -1.262 | **-1.484** | -1.385 | -1.064 | -1.358 | -1.219 | -1.311 |
| (5) NC-SV-d-r | -1.010 | -1.654 | -1.135 | -1.267 | -1.496 | -1.388 | -1.064 | -1.359 | -1.223 | -1.320 |
| (6) NC-SV-d-dy | -1.009 | -1.652 | -1.134 | -1.263 | -1.494 | -1.387 | -1.064 | -1.358 | -1.221 | -1.318 |
| (7) NC-SV-h-r | -1.003 | -1.646 | -1.129 | -1.258 | -1.488 | **-1.381** | -1.058 | **-1.350** | -1.215 | **-1.310** |
| (8) NC-SV-h-dy | -1.003 | -1.645 | -1.128 | **-1.258** | -1.488 | -1.382 | **-1.058** | -1.351 | -1.215 | -1.310 |
| (9) CC-RV-d-r | -1.014 | -1.644 | -1.137 | -1.266 | -1.489 | -1.389 | -1.069 | -1.363 | -1.223 | -1.315 |
| (10) CC-RV-d-dy | -1.012 | -1.643 | -1.132 | -1.265 | -1.489 | -1.386 | -1.068 | -1.361 | -1.221 | -1.314 |
| (11) CC-RV-h-r | -1.011 | -1.637 | -1.131 | -1.262 | -1.484 | -1.385 | -1.065 | -1.357 | -1.219 | -1.310 |
| (12) CC-RV-h-dy | -1.012 | -1.638 | -1.131 | -1.263 | -1.484 | -1.385 | -1.065 | -1.357 | -1.219 | -1.310 |
| (13) CC-SV-d-r | -1.010 | -1.654 | -1.139 | -1.265 | -1.496 | -1.389 | -1.066 | -1.359 | -1.224 | -1.319 |
| (14) CC-SV-d-dy | -1.009 | -1.655 | -1.134 | -1.266 | -1.494 | -1.387 | -1.064 | -1.359 | -1.222 | -1.317 |
| (15) CC-SV-h-r | **-1.003** | -1.644 | -1.129 | -1.259 | -1.488 | -1.382 | -1.058 | -1.351 | -1.215 | -1.310 |
| (16) CC-SV-h-dy | -1.003 | -1.646 | **-1.128** | -1.258 | -1.488 | -1.382 | -1.058 | -1.352 | **-1.215** | -1.311 |
| (17) L-RV-s-r | -1.012 | **-1.637** | -1.132 | -1.263 | -1.484 | -1.386 | -1.066 | -1.358 | -1.220 | -1.312 |
| (18) L-RV-s-dy | -1.014 | -1.639 | -1.132 | -1.265 | -1.485 | -1.385 | -1.068 | -1.357 | -1.220 | -1.313 |
| (19) L-SV-s-r | -1.004 | -1.650 | -1.129 | -1.260 | -1.491 | -1.385 | -1.061 | -1.351 | -1.217 | -1.312 |
| (20) L-SV-s-dy | -1.005 | -1.649 | -1.131 | -1.261 | -1.489 | -1.384 | -1.061 | -1.353 | -1.216 | -1.311 |

Table 4: Average log predictive scores for 20 assets for 20 models (sample: 02.01.2002 - 29.12.2006)

| | AIG | BA | BAC | C | F | GE | GS | HD | HPQ | IBM |
|---|---|---|---|---|---|---|---|---|---|---|
| (1) NC-RV-d-r | -1.473 | -1.564 | -1.195 | -1.366 | -1.802 | -1.362 | -1.581 | -1.570 | -1.756 | -1.333 |
| (2) NC-RV-d-dy | -1.463 | -1.567 | -1.188 | -1.355 | -1.799 | -1.353 | -1.577 | -1.566 | -1.750 | -1.325 |
| (3) NC-RV-h-r | -1.469 | -1.560 | -1.188 | -1.359 | -1.795 | -1.355 | -1.576 | -1.564 | -1.748 | -1.326 |
| (4) NC-RV-h-dy | -1.470 | -1.560 | -1.188 | -1.358 | -1.795 | -1.354 | -1.575 | -1.563 | -1.748 | -1.326 |
| (5) NC-SV-d-r | -1.485 | -1.559 | -1.207 | -1.388 | -1.810 | -1.356 | -1.583 | -1.568 | -1.774 | -1.330 |
| (6) NC-SV-d-dy | -1.475 | -1.562 | -1.202 | -1.377 | -1.808 | -1.352 | -1.582 | -1.563 | -1.770 | -1.328 |
| (7) NC-SV-h-r | -1.479 | -1.555 | -1.201 | -1.382 | -1.803 | -1.348 | -1.580 | -1.558 | -1.767 | -1.325 |
| (8) NC-SV-h-dy | -1.480 | -1.555 | -1.199 | -1.379 | -1.803 | -1.350 | -1.578 | **-1.558** | -1.767 | -1.324 |
| (9) CC-RV-d-r | -1.468 | -1.562 | -1.192 | -1.363 | -1.801 | -1.361 | -1.579 | -1.566 | -1.755 | -1.334 |
| (10) CC-RV-d-dy | **-1.459** | -1.566 | -1.186 | **-1.352** | -1.799 | -1.356 | -1.578 | -1.564 | -1.751 | -1.327 |
| (11) CC-RV-h-r | -1.468 | -1.559 | -1.187 | -1.358 | -1.795 | -1.354 | -1.575 | -1.563 | -1.748 | -1.326 |
| (12) CC-RV-h-dy | -1.468 | -1.560 | -1.186 | -1.355 | **-1.795** | -1.354 | **-1.575** | -1.564 | **-1.748** | -1.326 |
| (13) CC-SV-d-r | -1.481 | -1.558 | -1.206 | -1.390 | -1.812 | -1.360 | -1.583 | -1.564 | -1.774 | -1.336 |
| (14) CC-SV-d-dy | -1.471 | -1.563 | -1.201 | -1.381 | -1.807 | -1.352 | -1.585 | -1.559 | -1.769 | -1.330 |
| (15) CC-SV-h-r | -1.478 | -1.555 | -1.200 | -1.378 | -1.803 | -1.350 | -1.578 | -1.558 | -1.766 | -1.324 |
| (16) CC-SV-h-dy | -1.478 | -1.555 | -1.200 | -1.384 | -1.803 | **-1.345** | -1.580 | -1.558 | -1.767 | -1.325 |
| (17) L-RV-s-r | -1.469 | -1.559 | -1.188 | -1.359 | -1.796 | -1.355 | -1.576 | -1.564 | -1.749 | -1.326 |
| (18) L-RV-s-dy | -1.464 | -1.561 | **-1.182** | -1.355 | -1.796 | -1.356 | -1.576 | -1.565 | -1.752 | -1.325 |
| (19) L-SV-s-r | -1.479 | **-1.554** | -1.204 | -1.380 | -1.804 | -1.347 | -1.585 | -1.565 | -1.768 | -1.325 |
| (20) L-SV-s-dy | -1.468 | -1.556 | -1.193 | -1.375 | -1.804 | -1.355 | -1.581 | -1.559 | -1.768 | **-1.324** |

| | JNJ | JPM | KO | MO | NKE | PFE | PG | VZ | WMT | XOM |
|---|---|---|---|---|---|---|---|---|---|---|
| (1) NC-RV-d-r | -1.111 | -1.513 | -1.197 | -1.365 | -1.430 | -1.453 | -1.122 | -1.463 | -1.342 | -1.398 |
| (2) NC-RV-d-dy | -1.100 | -1.506 | -1.189 | -1.366 | -1.428 | -1.450 | -1.121 | -1.457 | -1.335 | -1.395 |
| (3) NC-RV-h-r | -1.106 | -1.505 | -1.190 | -1.364 | -1.424 | -1.448 | -1.114 | -1.457 | -1.335 | -1.392 |
| (4) NC-RV-h-dy | -1.106 | -1.504 | -1.190 | -1.364 | -1.425 | -1.448 | -1.114 | -1.458 | -1.335 | -1.392 |
| (5) NC-SV-d-r | -1.098 | -1.520 | -1.199 | -1.369 | -1.438 | -1.466 | -1.115 | -1.457 | -1.344 | -1.390 |
| (6) NC-SV-d-dy | -1.094 | -1.517 | -1.197 | -1.369 | -1.437 | -1.462 | -1.114 | -1.456 | -1.337 | -1.385 |
| (7) NC-SV-h-r | -1.089 | -1.512 | -1.192 | -1.366 | -1.430 | -1.459 | -1.107 | -1.452 | -1.334 | **-1.379** |
| (8) NC-SV-h-dy | -1.089 | -1.510 | -1.190 | -1.363 | -1.431 | -1.459 | **-1.107** | -1.452 | -1.334 | -1.380 |
| (9) CC-RV-d-r | -1.111 | -1.510 | -1.197 | -1.366 | -1.427 | -1.452 | -1.119 | -1.465 | -1.340 | -1.396 |
| (10) CC-RV-d-dy | -1.103 | -1.508 | **-1.188** | -1.369 | -1.430 | -1.451 | -1.124 | -1.458 | -1.336 | -1.392 |
| (11) CC-RV-h-r | -1.106 | **-1.504** | -1.190 | -1.363 | **-1.424** | -1.448 | -1.114 | -1.458 | -1.334 | -1.391 |
| (12) CC-RV-h-dy | -1.108 | -1.504 | -1.190 | -1.365 | -1.425 | -1.448 | -1.117 | -1.457 | -1.336 | -1.390 |
| (13) CC-SV-d-r | -1.098 | -1.515 | -1.203 | -1.374 | -1.437 | -1.467 | -1.113 | -1.459 | -1.343 | -1.389 |
| (14) CC-SV-d-dy | -1.090 | -1.521 | -1.195 | -1.368 | -1.437 | -1.464 | -1.115 | -1.456 | -1.337 | -1.384 |
| (15) CC-SV-h-r | -1.089 | -1.507 | -1.191 | -1.363 | -1.431 | -1.459 | -1.107 | -1.452 | -1.333 | -1.380 |
| (16) CC-SV-h-dy | **-1.089** | -1.512 | -1.191 | -1.366 | -1.431 | -1.459 | -1.107 | -1.454 | -1.333 | -1.381 |
| (17) L-RV-s-r | -1.106 | -1.505 | -1.191 | -1.364 | -1.425 | -1.449 | -1.115 | -1.457 | -1.335 | -1.392 |
| (18) L-RV-s-dy | -1.105 | -1.506 | -1.190 | -1.367 | -1.425 | **-1.447** | -1.115 | -1.455 | **-1.332** | -1.392 |
| (19) L-SV-s-r | -1.090 | -1.514 | -1.193 | -1.363 | -1.432 | -1.461 | -1.109 | -1.450 | -1.335 | -1.381 |
| (20) L-SV-s-dy | -1.089 | -1.513 | -1.194 | **-1.362** | -1.433 | -1.459 | -1.108 | **-1.450** | -1.333 | -1.381 |

Table 5: Average log predictive scores for 20 assets for 20 models (sample: 03.01.2007 - 31.12.2012)

| | AIG | BA | BAC | C | F | GE | GS | HD | HPQ | IBM |
|---|---|---|---|---|---|---|---|---|---|---|
| (1) NC-RV-d-r | -2.180 | -1.580 | -2.037 | -2.062 | -2.017 | -1.600 | -1.857 | -1.607 | -1.610 | -1.258 |
| (2) NC-RV-d-dy | -2.177 | -1.582 | -2.037 | -2.060 | -2.014 | -1.599 | -1.856 | -1.603 | -1.614 | -1.254 |
| (3) NC-RV-h-r | -2.174 | -1.576 | -2.031 | -2.056 | -2.010 | -1.594 | -1.850 | -1.600 | -1.604 | -1.252 |
| (4) NC-RV-h-dy | -2.175 | **-1.576** | -2.031 | -2.056 | -2.010 | -1.593 | -1.850 | -1.600 | -1.604 | -1.252 |
| (5) NC-SV-d-r | -2.188 | -1.596 | -2.033 | -2.069 | -2.027 | -1.593 | -1.867 | -1.614 | -1.612 | -1.262 |
| (6) NC-SV-d-dy | -2.180 | -1.595 | -2.035 | -2.067 | -2.020 | -1.595 | -1.866 | -1.611 | -1.615 | -1.256 |
| (7) NC-SV-h-r | -2.181 | -1.590 | -2.029 | -2.061 | -2.019 | -1.587 | -1.859 | -1.605 | -1.605 | -1.254 |
| (8) NC-SV-h-dy | -2.182 | -1.590 | -2.028 | -2.061 | -2.018 | **-1.587** | -1.858 | -1.605 | -1.605 | -1.254 |
| (9) CC-RV-d-r | -2.180 | -1.581 | -2.038 | -2.063 | -2.019 | -1.600 | -1.858 | -1.606 | -1.609 | -1.257 |
| (10) CC-RV-d-dy | -2.179 | -1.584 | -2.039 | -2.066 | -2.014 | -1.602 | -1.855 | -1.605 | -1.618 | -1.257 |
| (11) CC-RV-h-r | -2.174 | -1.576 | -2.031 | **-2.056** | -2.010 | -1.595 | -1.851 | **-1.600** | **-1.604** | -1.252 |
| (12) CC-RV-h-dy | -2.176 | -1.576 | -2.032 | -2.060 | **-2.010** | -1.594 | -1.851 | -1.600 | -1.606 | **-1.251** |
| (13) CC-SV-d-r | -2.190 | -1.597 | -2.038 | -2.071 | -2.027 | -1.600 | -1.867 | -1.613 | -1.612 | -1.264 |
| (14) CC-SV-d-dy | -2.175 | -1.598 | -2.037 | -2.069 | -2.018 | -1.596 | -1.862 | -1.611 | -1.616 | -1.257 |
| (15) CC-SV-h-r | -2.181 | -1.589 | **-2.027** | -2.061 | -2.019 | -1.589 | -1.858 | -1.605 | -1.605 | -1.254 |
| (16) CC-SV-h-dy | -2.181 | -1.590 | -2.029 | -2.062 | -2.018 | -1.590 | -1.857 | -1.606 | -1.606 | -1.254 |
| (17) L-RV-s-r | **-2.173** | -1.577 | -2.031 | -2.056 | -2.011 | -1.594 | **-1.850** | -1.600 | -1.604 | -1.252 |
| (18) L-RV-s-dy | -2.198 | -1.576 | -2.036 | -2.060 | -2.010 | -1.595 | -1.851 | -1.600 | -1.605 | -1.253 |
| (19) L-SV-s-r | -2.183 | -1.590 | -2.029 | -2.061 | -2.018 | -1.588 | -1.858 | -1.612 | -1.605 | -1.255 |
| (20) L-SV-s-dy | -2.195 | -1.590 | -2.033 | -2.065 | -2.017 | -1.588 | -1.858 | -1.605 | -1.606 | -1.256 |

| | JNJ | JPM | KO | MO | NKE | PFE | PG | VZ | WMT | XOM |
|---|---|---|---|---|---|---|---|---|---|---|
| (1) NC-RV-d-r | -0.9344 | -1.842 | -1.101 | -1.207 | -1.570 | -1.372 | -1.051 | -1.322 | -1.167 | -1.347 |
| (2) NC-RV-d-dy | -0.9336 | -1.843 | -1.095 | -1.204 | -1.566 | -1.372 | -1.055 | -1.323 | -1.168 | -1.344 |
| (3) NC-RV-h-r | -0.9292 | -1.838 | -1.094 | -1.201 | -1.562 | -1.368 | -1.048 | -1.315 | -1.164 | -1.342 |
| (4) NC-RV-h-dy | -0.9294 | -1.838 | -1.093 | -1.201 | -1.562 | -1.368 | -1.047 | -1.316 | -1.164 | -1.342 |
| (5) NC-SV-d-r | -0.9336 | -1.856 | -1.098 | -1.206 | -1.575 | -1.360 | -1.045 | -1.316 | -1.167 | -1.364 |
| (6) NC-SV-d-dy | -0.9337 | -1.855 | -1.094 | -1.203 | -1.573 | -1.363 | -1.048 | -1.315 | -1.165 | -1.361 |
| (7) NC-SV-h-r | -0.9263 | -1.849 | -1.090 | **-1.194** | -1.566 | -1.355 | -1.042 | **-1.304** | -1.159 | -1.353 |
| (8) NC-SV-h-dy | -0.9263 | -1.849 | -1.089 | -1.195 | -1.566 | -1.356 | **-1.041** | -1.306 | **-1.159** | -1.354 |
| (9) CC-RV-d-r | -0.9317 | -1.845 | -1.100 | -1.205 | -1.569 | -1.372 | -1.051 | -1.321 | -1.167 | -1.346 |
| (10) CC-RV-d-dy | -0.9290 | -1.847 | -1.094 | -1.204 | -1.568 | -1.367 | -1.051 | -1.320 | -1.164 | -1.347 |
| (11) CC-RV-h-r | -0.9287 | -1.839 | -1.094 | -1.201 | **-1.562** | -1.368 | -1.048 | -1.315 | -1.164 | **-1.341** |
| (12) CC-RV-h-dy | -0.9281 | -1.839 | -1.093 | -1.202 | -1.562 | -1.367 | -1.047 | -1.315 | -1.162 | -1.342 |
| (13) CC-SV-d-r | -0.9325 | -1.859 | -1.102 | -1.200 | -1.575 | -1.362 | -1.051 | -1.313 | -1.168 | -1.363 |
| (14) CC-SV-d-dy | -0.9352 | -1.860 | -1.094 | -1.206 | -1.573 | -1.361 | -1.047 | -1.315 | -1.168 | -1.361 |
| (15) CC-SV-h-r | **-0.9258** | -1.849 | -1.090 | -1.198 | -1.565 | **-1.355** | -1.042 | -1.305 | -1.159 | -1.353 |
| (16) CC-SV-h-dy | -0.9263 | -1.850 | **-1.089** | -1.195 | -1.566 | -1.356 | -1.041 | -1.306 | -1.159 | -1.355 |
| (17) L-RV-s-r | -0.9292 | **-1.837** | -1.095 | -1.201 | -1.562 | -1.369 | -1.048 | -1.316 | -1.164 | -1.342 |
| (18) L-RV-s-dy | -0.9339 | -1.839 | -1.094 | -1.201 | -1.563 | -1.369 | -1.051 | -1.317 | -1.166 | -1.343 |
| (19) L-SV-s-r | -0.9262 | -1.851 | -1.089 | -1.195 | -1.568 | -1.358 | -1.042 | -1.306 | -1.159 | -1.354 |
| (20) L-SV-s-dy | -0.9272 | -1.851 | -1.090 | -1.198 | -1.566 | -1.358 | -1.043 | -1.307 | -1.162 | -1.353 |