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Polarization or Moderation? Intra-group heterogeneity in endogenous-policy contests[☆]

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Abstract

We analyze the selection of a policy platform by a group of heterogeneous agents to confront the status quo policy defended by another group in a subsequent contest. This policy choice results from the interaction between the inter-group effects that lead to strategic restraint and the intra-group effects due to the heterogeneity among challengers. We detail the conditions that give rise to polarization or moderation of the selected challenging policy with respect to what would be selected by this group in the absence of any struggle.

Keywords: political processes; conflict; group contests; endogenous claims; intra-group heterogeneity
JEL Classification: D72, D74, C72

1. Introduction

The choice of the common policy platform by the members of a group in a political competition naturally displays a tension between selecting the policy that maximizes the probability of winning or selecting the most preferred one.[1] If, additionally, this group is composed of heterogeneous agents, an internal conflict among its members also comes into play. This occurs, for instance, when a political party internally chooses an alternative (either a policy or a candidate) to face the opponents' choice in a subsequent dispute and also in conflicts among industries, lobbies or interest groups. This study aims to shed some light on the effect of intra-group heterogeneity on this classical trade-off of the policy choice when the subsequent competition is modeled as a contest.

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[1] This is the main trade-off in Wittman's model (Wittman, 1977).

We consider the canonical environment of collective choice where the policy space is one-dimensional and agents have single-peaked preferences. These agents are organized into two groups: Defenders of the status quo and challengers. Without loss of generality, it is assumed that challengers prefer policies to the left of the policy space. These agents are heterogeneous regarding their most preferred policy or peak. Contrarily, status-quo defenders are homogeneous and they all prefer the extreme-right policy. A two-stage game is played: First, challengers set a common target-policy. Second, a contest between challengers and status-quo defenders determines which of the two policies is finally implemented. In our baseline model, the winning probability of a group is determined by a contest success function (hereafter CSF) that depends on the relative size of the group's aggregate effort. Agents select their effort individually and non-cooperatively and the cost-effort function is convex. For most part of the paper, we shall assume that the target-policy of the leftist group is selected by a representative. The alternative setting in which this policy is collectively selected by the group is also considered. In these cases, we focus on processes satisfying the *Condorcet Criterion*; *i.e.*, procedures that select the Condorcet winner when it exists.

Our analysis contributes to the literature on endogenous public policies and contests. Epstein and Nitzan (2004, 2007) show that in a contest to implement a one-dimensional policy among two groups of homogeneous agents, if individuals' preferences are strictly concave and differentiable then any group would have incentives to sacrifice some utility by moderating the target-policy because this lessens competition at the contest stage. This moderation is known in the literature as *strategic restraint*.[2] The intuition behind this result is that the concavity of preferences causes that a slight moderation of a group's target-policy reduces its members' stake just marginally but it leads to a first-order impact on the opponents' stake. This asymmetric effect on stakes translates into an increase of the winning probability of the conceding group, which overcomes the marginal utility loss of its members due to moderation.[3] If the group members have heterogeneous preferences, a slight moderation of a group target-policy may no longer entail a marginal decrease of the aggregate stake of the conceding group. So, it is not clear whether strategic restraint will still arise in our setting: A marginal moderation of the challenging policy causes a reduction of the stake of the status-quo defenders that leads them to lower their effort (inter-group effect), but it also implies a non-marginal modification of the aggregate stake of the challengers that leads them to vary their effort (intra-group effect). The resulting interaction between these two effects on the winning probabil-

[2]Münster (2006) extends Epstein's and Nitzan's (2004) analysis to an all-pay auction contest. Many papers also follow from that seminal paper to address environmental issues as Heyes (1997), Liston-Heyes (2001) or Friehe (2013). Strategic restraint is also studied in voting contexts with policy motivated candidates, e.g. Lindbeck and Weibull (1993).

[3]Cardona and Rubí-Barceló (2016) show that with tent-shaped or linear preferences, moderation is not obtained and groups would claim their most preferred policies.

ity, the winning utility and the cost of effort will determine the rise of moderation or polarization.

In order to disentangle the intra- and inter-group forces that affect the strategic choice of the target-policy, we first analyze the internal effect in isolation. To this end, we analyze situations where the target policy selected by the challengers has no effect on the aggregate effort of the status-quo defenders. Hence, such a choice would affect the winning probability only through the efforts of the challengers.[4] Note that having non-strategic opponents in the setting of Epstein and Nitzan (2004) would imply that there is nothing to win from moderation; so, parties would stick to their preferred policy. Hence, in this setting (with either non-strategic opponents or asymmetric information), the strategic choice of the target-policy of the challenging group would exclusively respond to *intra-group* heterogeneity. This heterogeneity induces the trade-off between utility and winning probability mentioned previously, which makes the optimal target-policy of any challenging agent lying in between her most-preferred policy and that one maximizing the aggregate effort of the group: Agents reduce their claims in exchange for a larger winning probability. Thus, intra-group forces push any representative to select a target-policy more moderated than her most preferred policy only if the latter is sufficiently polarized (more than the policy maximizing the aggregate effort). Otherwise, the intra-group effect yields polarization. These results are extended to situations where this policy is collectively selected by the group when preferences are quadratic. In these cases, there is a Condorcet-winner policy, which coincides with the optimal target-policy of the player with the median peak in the group.

When the effort of defenders in the contest depend on the target-policy selected by the challenging group, both inter- and intra-group effects would determine the policy choice. In these cases, we show that moderation is always profitable. That is, any agent in the challenging group prefers a target-policy lying in between the status quo and her most preferred policy. Therefore, the positive inter-group effects of moderation dominate any possible negative intra-group effect. This also occurs when the target-policy collectively selected. In these cases we show that under quadratic preferences, if the Condorcet-winner policy exists then it cannot be more polarized than the median's peak. Interestingly, this result is independent of the groups' sizes. Thus, although moderation might reduce the winning utility of many challengers and increase the utility of a few status-quo defenders, a less extreme challenging policy is always profitable.

Finally, our results are extended into two directions: First, we consider linear (instead of convex) costs of effort. Unlike the main framework, in this setting non-extreme agents will be non-active in the contest; but results are essentially the

[4] One might interpret this situation either as one in which the defenders do not choose their effort strategically, or as the situation where the target-policy is internally decided and not observed by the status-quo defenders, as in Nitzan and Ueda (2016).

same. Second, we study an alternative CSF, the linear-difference form, to show how some of our results crucially depend on this element. In particular, we show that polarization might arise from the interaction between intra- and inter-group forces when the most preferred policy of the representative is sufficiently moderate and the number of status-quo defenders is low enough to limit the inter-group benefits of a policy moderation. This result also extends to the case in which the target-policy is collectively selected.

As the choice of a particular policy entails an internal distribution of efforts and therefore utilities, in our model the selected target-policy can be interpreted as a sharing rule, which certainly affects the aggregate effort of the group. From this viewpoint, our study can be related to the literature analyzing the effects of the internal sharing rule on the outcome of contests (e.g., Nitzan and Ueda, 2011 and 2016; Kolmar and Wagener, 2013 or Balart *et al.*, 2016). However, we differ from this literature because in our case the choice is less flexible. In another respect, the selection of a target-policy can be interpreted as the choice of a representative like in the context of delegation (Baik and Kim, 1997, Schoonbeek, 2004 or Baik, 2007) but in our framework delegation is partial: The delegate sets the target-policy but efforts are individually chosen by the members of the group.

The two-stage selection of policy and effort of our setting is closely related to valence models of political competition (Groseclose, 2001; Aragonès and Palfrey, 2002; Aragonès and Xefteris, 2012). In particular, to those where valence is endogenously determined (Hirsch, 2016; Ashworth and Bueno de Mesquita, 2009; Herrera *et al.*, 2008; Meirowitz, 2008; Schofield, 2006). We could interpret total effort exerted in the contest as 'campaign valences', in the sense of Carter and Patty (2015), which increase party's probability of winning the election. However, the sources of polarization or moderation of policy platforms present in those papers differ from ours. The first main difference is that polarization in our setting is a consequence of intra-party forces. Heterogeneity among party members is not considered in this specific literature, where a candidate rather than a party selects the policy platform. Only Schofield (2006) recognizes the role activists have in pulling equilibrium policy toward the extreme, without explicitly accounting for the distribution of party members. Concerning the inter-party forces, still many other differences can be found. In our setting, groups are policy motivated parties *à la Wittman*, in opposition to the Downsian party approach in Ashworth and Bueno de Mesquita (2009) and Meirowitz (2008). In our paper moderation is a consequence of the strategic interaction of effort choices in the contest game (strategic restraint), rather than a consequence of more or less uncertainty in elections (Wittman, 1983 and Calvert, 1985). In a setting with ideological voters, Herrera *et al.* (2008) also found that a policy moderation decreases campaigning costs and increments the winning probability. However, this increment comes from narrowing the gap between the proposed policy and the expected median voter position, and not from an strategic interaction with the opponent.

The next section presents the basic model and the equilibrium when the effort level of the status quo defenders is either fixed or endogenous. In Section 3, we discuss the policy choice when this is made collectively. In Section 4 we analyze the results under two alternative specifications: a constant marginal cost of effort and a CSF of linear-difference form. Section 5 concludes.

2. The model

A set of players N with cardinality n must choose the target-policy $x \in [0, 1]$ to compete against the status quo $y = 1$, defended by the m members of group M. Preferences of agent j over public policies are represented by $u_j(x) = 1 - \theta(|x - j|)$, where, with some abuse of notation, j denotes both the *peak* and an agent with that peak. Moreover, this function satisfies $\theta(0) = 0$, $\theta'(0) = 0$ and, for $z > 0$, $\theta'(z) > 0$ and $\theta''(z) > 0$. We assume that the members of N are (possibly) heterogeneous and have their peaks in $[0, 1/2]$ whereas all the members of M are identical and have their peak at 1. Some of our results are obtained under the specification $\theta(|x - j|) = (x - j)^2$. In this case, preferences are said to be *quadratic*.

Once the target-policy x has been settled, group N exerts an aggregate effort A to increase the probability of implementing x in the contest against the status quo. Let B denote the aggregate effort exerted by group M to defend the status quo. We consider a linear impact function, so that A is the result of adding up all the individual efforts a_j for all $j \in N$. Similarly, $B = \sum_{j \in M} b_j$, where b_j denotes the individual effort of an agent $j \in M$. The probability that policy x is implemented depends on the two aggregate efforts according to the function $p(A, B)$, referred as the contest success function (CSF henceforth). With the complementary probability, the status quo $y = 1$ remains. The CSF is assumed to be homogeneous of degree 0 so that $p(A, B) = f(A/B)$, satisfying (additionally) $f' > 0$ and $f'' < 0$. Efforts are costly. Particularly, the cost function is assumed to be quadratic. Agents' preferences over public policies and costs of effort are assumed to be separable so that the utility function can be written as

$$v_j(x, a_j) = u_j(x) - a_j^2/2.$$

To determine the policy selected by the group we solve the game backwards. First, for any $x \in [0, 1]$, we find the Nash equilibrium of the contest game, where agents select their efforts individually. Second, we determine the target-policy selected by the group. For now, we assume that the group has a representative $r \in N$ who has full authority to select such a policy. Let x_r^* denote this policy, that is, the optimal target-policy selected by agent r in response to the best reply of the agents in the second stage of the game. In Section 3, we consider that the target-policy selected by the group is the result of some internal collective decision process that satisfies the Condorcet criterion.

As in Epstein and Nitzan (2004), this setting will allow to analyze how the existence of a subsequent contest affects the strategic choice of a public policy (inter-group effect). But additionally, this choice can also be affected by the heterogeneity of the group (intra-group effect), as the present framework aims to show. In order to isolate the latter effect from the former, we will start by setting a fixed $B = \bar{B}$.[5]

2.1. Non-strategic opponent (NSO)

In the contest stage, for any $x \in [0,1]$, agent $j \in N$ chooses a_j to maximize

$$\begin{aligned} v_j(x, a_j) &= f((A_{-j} + a_j)/\bar{B}) u_j(x) + \left[1 - f\left((A_{-j} + a_j)/\bar{B}\right)\right] u_j(1) - a_j^2/2 \\ &= f\left((A_{-j} + a_j)/\bar{B}\right) D_j(x) + u_j(1) - a_j^2/2 \end{aligned}$$

where $A_{-j} = A - a_j$ and $D_j(x) = |u_j(x) - u_j(1)|$ denotes the stake of agent j. Hence, the optimal effort level a_j^* satisfies

$$f'\left(\left(A_{-j} + a_j^*\right)/\bar{B}\right) \frac{1}{\bar{B}} D_j(x) - a_j^* \equiv 0. \tag{1}$$

Adding up this condition for all players in N, we implicitly obtain the equilibrium aggregate effort of group N, $A^* = A\left(x, \bar{B}\right)$, as

$$f'\left(A^*/\bar{B}\right) \frac{1}{\bar{B}} D_N(x) - A^* \equiv 0. \tag{2}$$

where $D_N(x) = \sum_{j \in N} D_j(x)$.[6]

Note that, in equilibrium

$$a_j^*/A^* = \frac{D_j(x)}{D_N(x)} \text{ and } a_j^*/a_i^* = \frac{D_j(x)}{D_i(x)} \tag{3}$$

for all $i, j \in N$. Due to the strict convexity of effort costs, all agents will exert a positive effort. In this sense, there are no strong free-riders. Nevertheless, exerting effort generates positive externalities on the other members of the group.

Defining $\bar{Q}(x) = A^*/\bar{B}$,[7] condition (2) can be rewritten as

$$A^* = f'\left(\bar{Q}(x)\right)\left(1/\bar{B}\right) D_N(x)$$

[5] An alternative interpretation of this model with a fixed $B = \bar{B}$ is a standard public good provision where $p\left(A, \bar{B}\right)$ is the 'size' of the public good and $u_j(x) - u_j(1)$ is the j's marginal valuation of this public good when located at x. As commented in the Introduction, this setting also adapts to a context where only the members of N observe the target policy.

[6] As $f''\left(A/\bar{B}\right) < 0$ it is immediate that the solution is unique.

[7] Indeed, we have that $A\left(x, \bar{B}\right)$ and $\bar{Q}\left(x, \bar{B}\right)$ but we omit the parameter $\bar{B}$ to simplify the exposition.

and therefore

$$\bar{Q}(x) = f'\left(\bar{Q}(x)\right)\frac{1}{\bar{B}^2}D_N(x). \tag{4}$$

Differentiating (4) with respect to x yields

$$f''\left(\bar{Q}(x)\right)\frac{1}{\bar{B}^2}D_N(x)\bar{Q}'(x) + f'\left(\bar{Q}(x)\right)\frac{1}{\bar{B}^2}D'_N(x) - \bar{Q}'(x) = 0,$$

so that

$$\begin{aligned}\bar{Q}'(x) &= \frac{\bar{Q}(x)}{1 - f''\left(\bar{Q}(x)\right)\left(1/\bar{B}^2\right)D_N(x)}\frac{D'_N(x)}{D_N(x)}\\ &= \frac{f'\left(\bar{Q}(x)\right)\bar{Q}(x)}{f'\left(\bar{Q}(x)\right) - f''\left(\bar{Q}(x)\right)\bar{Q}(x)}\frac{D'_N(x)}{D_N(x)}.\end{aligned}$$

As $f'(Q) > 0$ and $f''(Q) \leq 0$, it can be concluded that

$$\bar{Q}'(x) \geq 0 \Longleftrightarrow D'_N(x) \geq 0.$$

Note that $D_N(x)$ is a sum of concave functions in x, which is (uniquely) maximal at $\overline{x}$ satisfying $\sum_{j\in N} u'_j(\overline{x}) = 0$.[8] Therefore, moving the target-policy towards $\overline{x}$ increases both the aggregate effort of group N and, given that M is non-strategic, its winning probability.

In addition to the effect on the expected utility of any agent $j \in N$ transmitted via the winning probability, there are two more aspects that influence the optimal target-policy choice of any particular representative. One is the direct utility she would obtain from the implementation of that policy; and the other is the cost of the effort she would exert in (the equilibrium of) the subsequent contest. The following proposition shows that the interaction among these three forces leads the optimal policy of any representative $r \in N$, x_r^*, to lie in between her peak r and $\overline{x}$. All proofs are in the Appendix.

Proposition 1. *If $r \neq \overline{x}$ then $(x_r^* - r)(x_r^* - \overline{x}) < 0$. Otherwise, $x_r^* = r = \overline{x}$.*

The intra-group forces caused by the heterogeneity of group N push any representative to move the target-policy away from her peak r towards $\overline{x}$ because this generates two positive effects on her that overcome the reduction of her gain from winning the contest. These positive effects are: (i) a cost-effort saving because a_r^* decreases as x moves from r towards $\overline{x}$,[9] and (ii) a stronger incentive of her

[8] For quadratic preferences $\overline{x}$ is the mean of the peaks.

[9] As $f''(Q) < 0$, it is immediate from (1) that $Q'(x) > 0$ and $D'_r(x) < 0$ imply that a_r^* must also decrease in x when moving from r towards $\overline{x}$, for any $r \in N$.

group members' to engage in rent-seeking efforts and, consequently, a higher winning probability. Thus, depending on the relative position of r with respect to $\overline{x}$, the target-policy of group N might differ from what would be selected in the absence of the subsequent contest; *i.e.*, the peak of its representative. Specifically, the existence of the contest stage would induce moderation when $r < \overline{x}$ and polarization when $r > \overline{x}$.

Note that if the members of group M condition strategically their efforts on the target-policy selected by group N, these intra-group forces would interact with the inter-group effect that is analyzed in Epstein and Nitzan (2004) or Cardona and Rubí-Barceló (2016). We address the analysis of this interaction next.

2.2. A strategic opponent (SO)

When B is strategically selected by the members of group M who observe the target-policy selected by group N, then the choice of this policy by the representative would balance the intra-group forces described above and the following inter-group effect: A moderation of the target-policy of group N would decrease the stake of the opponents and this would reduce their incentives to exert effort in the contest. Therefore, when $r < \overline{x}$ intra and inter-group forces are aligned to induce a moderation of the target-policy with respect to the representative's peak r. Otherwise, when $r > \overline{x}$ these two forces affect the optimal target-policy choice in opposite directions: The inter-group effect would move this policy towards moderation whereas the intra-group effects would induce polarization. The analysis of this conflict will focus our attention throughout this section.

For any $x \in [0,1]$, any agent $j \in M$ chooses an effort b_j to maximize

$$v_j(x, b_j) = \left[1 - f\left(A/\left(B_{-j} + b_j\right)\right)\right] D_j\left(x\right) + u_j\left(x\right) - b_j^2/2,$$

where $B_{-j} = B - b_j$. Hence, the optimal individual effort b_j^* satisfies

$$f'\left(A/\left(B_{-j} + b_j^*\right)\right) \frac{A}{\left(B_{-j} + b_j^*\right)^2} D_j\left(x\right) - b_j^* \equiv 0.$$

Aggregating for all agents $j \in M$, we implicitly obtain the effort B, as

$$f'\left(A/B\right) \frac{A}{B^2} D_M\left(x\right) - B \equiv 0. \tag{5}$$

where $D_M(x) = \sum_{j \in M} D_j\left(x\right)$. From (1) (but with a non-fixed B) and (3), a_j^* can be expressed as a function of $Q\left(x\right) \equiv A\left(x\right)/B\left(x\right)$ as follows

$$a_j^* = a_j\left(x\right) = \left[f'\left(Q\left(x\right)\right) Q\left(x\right) D_j\left(x\right) \frac{D_j\left(x\right)}{D_N\left(x\right)}\right]^{1/2}, \tag{6}$$

and from (2) (but with a non-fixed B) and (5), the same can be done for equilibrium efforts A^* and B^*, as follows

$$A^* = A(x) = \left[f'(Q(x))\,Q(x)\,D_N(x)\right]^{1/2}, \tag{7}$$

$$B^* = B(x) = \left[f'(Q(x))\,Q(x)\,D_M(x)\right]^{1/2}. \tag{8}$$

From these two expressions the following is obtained:

$$Q(x) = \left(\frac{D_N(x)}{D_M(x)}\right)^{1/2}. \tag{9}$$

This puts forward the importance of group sizes in determining the winning probability of the challengers. Nevertheless, moderating the target-policy increases this probability independently of group sizes, as the next result shows.

Lemma 1. $Q'(x) > 0$.

When a moderation of the representative agent r involves to bring the target-policy closer to $\overline{x}$, this moderation induces a higher aggregate effort of the challengers (intra-group effect) and a lower aggregate effort of the status-quo defenders (inter-group effect), that unequivocally increase the winning probability of group N. However, when a moderation involves to move the target-policy away from $\overline{x}$, the aggregate effort of both groups will be reduced. The previous lemma shows that, in this case, the positive inter-group effect of a target-policy moderation on the winning probability of group N offsets the negative intra-group effect. And this holds even for n/m arbitrarily large, so that this moderation increases the winning probability of the challenging policy even if the number of agents that originate the (possibly) negative intra-group effect is much higher than the number of agents causing the inter-group effect.

In general, the representative's incentives to moderate the target-policy should be interpreted as a consequence of the interaction among the following three effects. By moving the target-policy, the representative affects (i) her utility from winning the subsequent contest, (ii) as the stakes are also modified, all agents' incentives to engage in rent-seeking efforts and, consequently, the winning probability of her group, and (iii) her effort exerted in the contest. Lemma 1 shows that effect (ii) is always positive in case of a moderation. Moreover, regarding effect (i), the representative's utility from winning the contest is increased by a moderation if this implies moving the target-policy towards the representative's peak and reduced otherwise. The interaction of those two effects with (iii) determines the relative location of the optimal target-policy of the representative x_r^*, as specified next.

Proposition 2. $x_r^* > \min\{\overline{x}, r\}$. *Moreover, under quadratic preferences* $x_r^* > r$.

The proof of this Proposition shows that when $x \leq \min\{\overline{x}, r\}$, the two positive effects (i) and (ii) of a target-policy moderation from x are not offset by (iii). However, when $x \in [\overline{x}, r]$ this can only be proved under quadratic preferences: For the general set of preferences, a target-policy moderation might involve a negative effect (iii) sufficiently high to disincentivize moderation.[10] Finally, when $x \in [r, \overline{x}]$ a target-policy moderation implies that effect (i) is negative, so this moderation will only take place if this effect is sufficiently small.

This result reinforces Epstein and Nitzan (2004) as strategic restraint still arises after adding the forces caused by intra-group heterogeneity to the inter-group effects analyzed in that paper. Moreover, this is independent of the groups' sizes. When the representative is relatively extreme ($r < \overline{x}$), both intra and inter-group effects induce moderation. Thus, the optimal target-policy would be unequivocally larger than the representative's peak. Nevertheless, when the representative is relatively moderated ($r > \overline{x}$), intra-group forces would lead to polarization, so in that case the interaction with an strategic opponent is key for having moderation.

The comparison between the NSO and the SO cases is not direct because it would crucially depend on the exogenous value $\bar{B}$. A reasonable $\bar{B}$ for comparison purposes would satisfy the consistency requirement used in Nitzan and Ueda (2016). According to this requirement, $\bar{B}$ would correspond to the addition of the best responses of group M's members to the equilibrium effort level $A^* = A(x^*, \bar{B})$, where x^* is the equilibrium policy choice made by the representative of N when she considers that group M members would not react to x, as if they were not able to observe it. Using this $\bar{B}$ in the NSO case, Table 1 displays some numerical examples illustrating this comparison. In these simulations preferences are assumed to be quadratic and the CSF has a Tullock form $p(A, B) = A/(A + B)$, $n = 21$, $r = 0.001$ and $\overline{x} = 0.2386$).[11]

		x_r^*	$P(x_r^*)$	$(A; B)$
$m = 15$	NSO	0.0824	0.4900	(1.7070; 1.7766)
	SO	0.2799	0.5554	(1.7312; 1.3859)
$m = 21$	NSO	0.0883	0.4501	(1.7017; 2.0786)
	SO	0.2990	0.5198	(1.7377; 1.6050)
$m = 25$	NSO	0.0913	0.4297	(1.6946; 2.2491)
	SO	0.3088	0.5013	(1.7371, 1.7280)

Table 1: NSO case vs. SO case ($n = 21$, $r = 0.001$, $\overline{x} = 0.2386$).

In all of them, there is always more moderation under the SO case. Moreover group N (M) exerts more (less) effort in the equilibrium of the SO case so that the

[10] We are not conclusive at this point, as we do not have any example where this happens. Although $a_r(x)/A(x)$ increases in those cases, it is usually the case that $a_r(x)$ decreases as well.

[11] It is worth noting that equilibria are fully characterized by the set of parameters $(n, m, \overline{x}, r)$.

winning probability of N is higher in this case. This suggests that it is in the interest of group N's members to make public their chosen target-policy if unobservable.

3. Collective choice of the target-policy

Until now, we considered an exogenously determined representative of group N selecting the target-policy of the group.[12] Now, we assume that the target-policy is collectively selected by the members of N following a process that satisfies the Condorcet Criterion.[13] The existence of a Condorcet-winner policy (denoted by x_w^*) would contribute a rationale for the assumption of the exogenous representative in our previous setting. This existence is guaranteed when all individuals have single-peaked preferences (*Median Voter Theorem*, Black, 1958) or when the utilities of any two individuals satisfy single-crossing (*Representative Voter Theorem*, Rothstein, 1991). Additionally, in the latter case, the Condorcet winner corresponds to the choice of the median player. Let d denote both the median of N (and her peak). Although we are not able to show that in our model agents' indirect utility functions satisfy single-peakedness, they satisfy single-crossing in the NSO case when preferences are quadratic. Consequently:

Proposition 3. *(NSO case) Under quadratic preferences, the optimal target-policy of the median player (x_d^*) is the Condorcet winner (x_w^*).*

Proposition 1 can be applied here to conclude that the interaction among intra and inter-group forces leads the target-policy of group N (*i.e.* the Condorcet-winner policy in this setting) to lie in between the peak of the median agent d and $\overline{x}$. So, when compared with the policy selected in the absence of the contest (d), there is moderation when $d < \overline{x}$ and polarization when $d > \overline{x}$.

Although we are not able to provide a general proof of existence of a Condorcet-winner policy in the SO case, we obtained a 'partial single-crossing' property (Lemma 5 in Appendix B) that allows us to characterize its relative location in case of existence.

Proposition 4. *(SO case) Under quadratic preferences, if a Condorcet-winner policy exists then it cannot be more polarized than x_d^*.*

From the second part of Proposition 2 we already know that under quadratic preferences the optimal target-policy for any player is more moderated than her peak,

[12] Nitzan and Ueda (2016) consider a representative that maximizes the aggregate surplus of the group.

[13] Condorcet winners are a robust prediction of the group's decision, particularly "for situations in which people can act in concert, with various subsets of people coordinating their actions to form coalitions [...] for unilaterally insuring an improvement in the welfare of all of its members"(Ordeshook, 1980).

so that $x_d^* > d$. Consequently, the last result implies that, whenever a Condorcet winner exists, the contest induces group N to collectively select a more moderated target-policy than what would be selected otherwise (d). As in the case with an exogenously given representative, the interaction between intra and inter-group forces would lead to strategic moderation.

In spite of not having a general proof of existence of a Condorcet-winner policy, in all our numerical simulations (i) indirect utilities are single-peaked and (ii) there is a positive monotonicity between agents' peaks and their optimal target policies. Table 2 displays some of these simulations in which preferences are assumed to be quadratic, the CSF has a Tullock form $p(A, B) = A/(A + B)$, $d = 1/4$ and $\overline{x} = 71/300$.

	$j = 0$	$j = 1/10$	$j = 1/4$	$j = 1/3$	$j = 1/2$
$n = 25, m = 2$	0.1600	0.2203	0.3254	0.3903	0.5304
$n = 15, m = 2$	0.1889	0.2436	0.3414	0.4028	0.5375
$n = 5, m = 2$	0.2682	0.3075	0.3848	0.4368	0.5570
$n = 5, m = 6$	0.3313	0.3637	0.4291	0.4741	0.5810

Table 2: x_j^* for different relative group sizes with $d = 1/4$ and $\overline{x} = 71/300$.

Notice that the optimal target-policy of any individual is more moderated than her peak, as announced by Proposition 2. Moreover, as the two properties mentioned above hold, the Condorcet-winner policy x_w^* exists and it is equal to x_d^*. Finally, this table illustrates that the magnitude of moderation does depend upon the relative group size: The smaller is n/m the larger will be the moderation of any agent j with respect to her peak, *i.e.* the larger is $x_j^* - j$. The reason is that, for a bigger M inter-group forces are stronger: The gains from moderating the challenging target-policy in terms of the winning probability are more prominent, as this moderation will cause a reduction of the effort exerted by more status-quo defenders.

4. Further specifications

In this section, we extend the model in two directions to analyze the robustness of our results. Specifically, these variations are considered: linear cost function and linear-difference CSF.

4.1. Linear costs

The case of linear costs is particularly illustrative because in these cases the total effort of a group depends only on the highest individual stake. In homogeneous groups, this means that the group effort is independent of the size of the group whereas in heterogeneous groups, this implies that only the extreme agents would exert a positive effort. Without loss of generality, we assume that there is an agent at 0. Hence, the effort of a group N would depend only on the stake of this extreme member(s). Specifically, A solves

$$f'(A/B)\frac{1}{B}D_0(x)-1=0. \tag{10}$$

Similarly, B solves

$$f'(A/B)\frac{A}{B^2}D_1(x)-1=0,$$

where D_1 is the stake of any member of M that, in our setting, is located at 1. From these two equations, we obtain that

$$Q(x)=\frac{A(x)}{B(x)}=\frac{D_0(x)}{D_1(x)}\Longrightarrow Q'(x)=Q(x)\left(\frac{D_0'(x)}{D_0(x)}-\frac{D_1'(x)}{D_1(x)}\right).$$

Thus, the optimal target-policy of any representative $r\in N$, $r\neq 0$, satisfies

$$\begin{aligned}0 &= f'(Q(x))\,Q'(x)\,D_r(x)+f(Q(x))\,D_r'(x)\\ &= f'(Q(x))\,Q(x)\left(\frac{D_0'(x)}{D_0(x)}-\frac{D_1'(x)}{D_1(x)}\right)D_r(x)+f(Q(x))\,D_r'(x)\,.\end{aligned}$$

The convexity of θ implies that $\frac{D_0'(x)}{D_0(x)}-\frac{D_1'(x)}{D_1(x)}>0$ for $x\leq 1/2$. Hence, as $r\leq 1/2$ this implies that $D_r'(x_r^*)<0$ so that $x_r^*>r$ for all $r\in N$.[14] That is, any representative of group N will choose a target-policy more moderated than her own peak.[15] As in the baseline model, the effect of the inter-group forces that push the representative to moderate the target-policy in order to increase the winning probability of her group predominate over the intra-group effects, which would lead to polarization.[16] The difference with respect to the quadratic-costs case is that now a moderation does not alter the cost of effort of any representative $r>0$, as all the effort will be exerted by the extreme member(s). Thus, for a target-policy moderation to be beneficial it suffices that the benefits from increasing the winning probability offset the utility loss from choosing a less preferred target-policy, as it actually happens.

In case that the target-policy of N is collectively selected by its members, the existence of a Condorcet-winner policy is not guaranteed, as in the case of quadratic

[14] If $r=0$ then this agent would possibly exert a positive effort. In these cases, the positive effects of moderation would be reinforced.

[15] Unlike the quadratic-cost case, now it can be proved that $x_r^*>r$ for the general set of preferences and not only for quadratic preferences.

[16] Specifically, when inter-group effects are neutralized, *i.e.* in the NSO case, $x_0^*=0$ and $x_r^*\in(0,r)$ for any $r\neq 0$ (see Appendix C). As in the main model, the intra-group forces arise from the trade-off faced by the representative between maximizing her utility in case of winning (at $x=r$) and maximizing the winning probability (at $x=0$, under linear costs).

costs. Again, we are able to obtain a 'partial single-crossing' property[17] that allows us to characterize the relative location of a potential Condorcet-winner policy. In particular, that property implies that x_d^* would be preferred to any $x < x_d^*$ for any agent $l \geq d$ when $d \neq 0$.[18] Therefore, if a Condorcet-winner policy exists then it cannot be more polarized than x_d^*, as in our main setting. Moreover, since $x_j^* > j$ for all j as showed above, it can be concluded that when a Condorcet-winner exists then it should be more moderated than the median's peak. Thus, considering linear costs of effort does change only slightly our qualitative results. The only remarkable difference with respect to our baseline model is that (as only extreme agents exert positive effort) single-crossing might fail because $x_0^* > x_j^*$ for some $j \in N$. This imply that, under linear costs of effort, the Condorcet-winner policy can be more moderated than x_d^* as illustrated by the examples displayed in Figure 1. In these examples, preferences are assumed to be quadratic, the CSF has a Tullock form $p(A, B) = A/(A + B)$ and $m = 1$.

4.2. Linear-difference Contest Success Function

We next consider situations where the winning probability of group N is given by $p(A, B) = 1/2 + s(A - B)$, for appropriate $s > 0$.[19] Cardona and Rubí-Barceló (2016) analyzed this setting for homogeneous groups and linear preferences and found that the equilibrium target policies were affected by the group size. In the present setting with non-linear preferences and heterogeneous groups, the choice of a specific target-policy would affect the agents' stakes differently, so the extension of those results is not immediate. In this section, the analysis focuses on the case of quadratic preferences.

By backwards induction, we start from the contest stage. There, the optimal effort level a_j^* satisfies

$$sD_j - a_j^* = 0.$$

Thus, $A^* = sD_N$. Then, the indirect utility function of a representative agent $r \in N$ is

$$V_r(x) = P(x)D_r(x) + u_r(1) - (sD_r(x))^2/2,$$

---

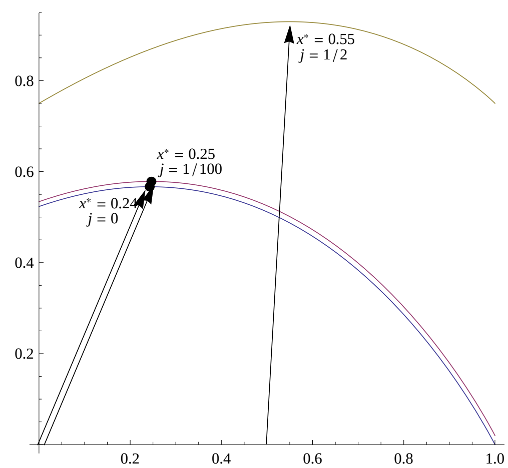
[17]See Appendix C.

[18]If $d = 0$ then it is immediate that the Condorcet-winner policy would be x_0^*.

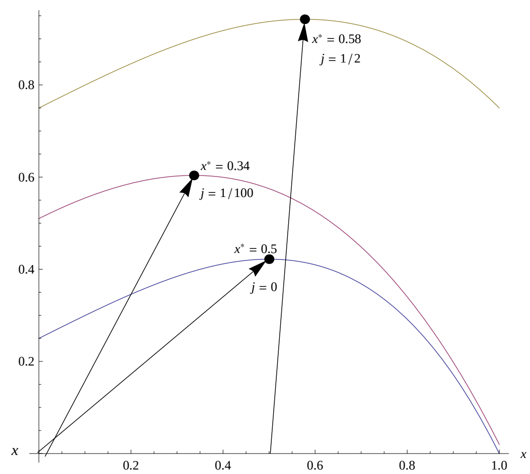
[19]In general, probabilities in the linear difference-form CSF are such that

$$p(A, B) = \max\{0, \min[1/2 + s(A - B), 1]\}.$$

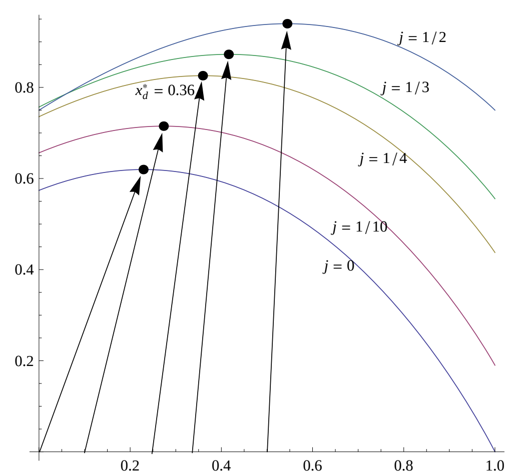
For some values of s, a pure-strategy equilibrium fails to exist. Che and Gale (2000) characterize the set of mixed strategy equilibria in contests between two players and linear costs of effort. To our knowledge there is no characterization of such equilibria in our setting. In this paper, we will focus on pure-strategy equilibria. Hence, restricting the parameter set would be required to guarantee existence.



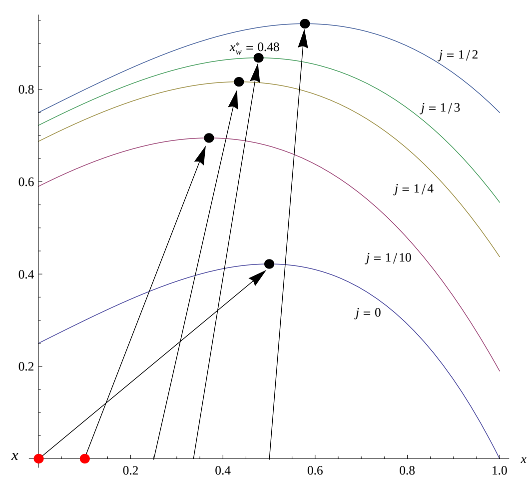
(a) Quadratic costs, $\overline{x} = 0.17$ and $n = 3$



(b) Linear costs, $\overline{x} = 0.17$ and $n = 3$



(c) Quadratic costs, $\overline{x} = 0.2367$, $n = 5$



(d) Linear costs, $\overline{x} = 0.2367$, $n = 5$

Figure 1: Quadratic vs. Linear costs ($m = 1$).

where $P(x) = p(A(x), B(x))$.[20] Following similar steps, we obtain that $B^* = sD_M$. Thus, $P = 1/2 + s^2(D_N - D_M)$ and $P' = s^2(D'_N - D'_M)$. Let $\hat{x} = \frac{n\overline{x}+m}{n+m}$. Then,

Proposition 5. *Under quadratic preferences and the linear-difference CSF, $x_r^* \geq r$ iff $r \leq \hat{x}$, for any $r \in N$.*

Unlike our previous results (obtained with a CSF homogeneous of degree zero), under the linear-difference CSF the representative will optimally choose a target-policy which is more moderated than her peak only when she is sufficiently extreme, *i.e.*, when $r < \hat{x}$. Otherwise, there will be polarization.[21] Notice that the threshold $\hat{x}$ is the weighted average between $\overline{x}$ and 1 (the two policies that maximize D_N and D_M, respectively) where weights are the relative sizes of each group. So, for a given r, the target-policy of N will be more polarized than its representative's peak only when $r > \overline{x}$ and the relative size of the opposite group (m) is so low that $r > \hat{x}$. Intuitively, in a contest against a sufficiently small group M, the positive inter-group effects of a target-policy moderation that lower the equilibrium effort of M are not enough to compensate the negative consequences of this moderation.

When the target-policy is collectively selected by the group members, the existence of a Condorcet-winner policy is guaranteed by the following result:

Lemma 2. *Under quadratic preferences and the linear-difference CSF, the indirect utility is single-peaked.*

This result guarantees the existence of a Condorcet winner policy x_w^* (*Median Voter Theorem*, Black, 1958).[22] The following result characterizes the location of this policy.

Proposition 6. *Under quadratic preferences and the linear-difference CSF, $x_w^* \geq d$ iff $d \leq \hat{x}$.*

In contrast with Proposition 4, under a linear-difference CSF the Condorcet-winner policy might be more polarized than the median's peak. This happens when such a median is sufficiently moderate; i.e., when $d > \hat{x}$. As the threshold $\hat{x}$ depends positively on the relative size of the opposite group,[23] the inter-group effect is positively affected by m/n. So, polarization is obtained only when the median is relatively moderated.

Although the general proof of existence of a Condorcet-winner policy in this case, single-crossing is not generally proved. In spite of that, in all our numerical

[20] Argument x will be omitted if no confusion arises.

[21] If only intra-group effects are considered, as in the NSO case, the result is similar but with $\overline{x}$ instead of $\hat{x}$ (see Appendix E).

[22] This result also holds for the NSO case (see Appendix E).

[23] In the NSO case, the result is similar but with $\overline{x}$ instead of $\hat{x}$ (see Appendix E).

simulations there is a positive monotonicity between agents' peaks and their optimal target policies, as in the main model. Table 3 displays some of these simulations in which preferences are assumed to be quadratic, the CSF is $p(A, B) = 1/2+s(A-B)$, $s = 1/4$, $d = 1/4$ and $\overline{x} = 71/300$.

	$j = 0$	$j = 1/10$	$j = 1/4$	$j = 1/3$	$j = 1/2$	$\hat{x}$
$n = 15, m = 2$	0.1745	0.2088	0.2796	**0.3311**	**0.4622**	0.3265
$n = 5, m = 2$	0.1987	0.2356	0.3103	0.3631	**0.4932**	0.4548
$n = 5, m = 10$	0.4665	0.4756	0.4974	0.5158	0.5753	0.7456

Table 3: x_j^* for different relative group sizes with $d = 1/4$, $s = 0.25$, and $\overline{x} = 71/300$.

Due to this monotonicity, the Condorcet-winner policy is equal to x_d^* in these cases. Notice that the optimal target-policy of any individual is more moderated than her peak only when $j < \hat{x}$ (non-bold cases in the table), as announced by Proposition 5. In the remaining cases, agent j would select a target-policy more polarized than her peak. Finally, this table illustrates that the magnitude of the moderation does depend upon the relative group size, as in the main setting.

5. Conclusion

We studied the contest between two groups of agents when one of them tries to modify the status quo. Previous to the contest challengers must set their target-policy. As showed in Epstein and Nitzan (2004), the choice of this target-policy affects the challengers' expected utility through three different channels: (i) their effort costs, (ii) their equilibrium winning probability and (iii) their equilibrium winning utility. The novelty is that in our setting part of these effects are due to the heterogeneity among challengers: We showed that when the probability of implementing the selected policy depends only on the efforts of the group, so the strategic effect acting in Epstein and Nitzan (2004) is neutralized, then the choice of the optimal target-policy solves the trade-off between maximizing the winning utility of the representative (at her most preferred policy) and maximizing her winning probability (at the policy where the aggregate stake of the group in the contest is maximized).[24] As a result, the relative location of these two policies will determine whether the optimal policy will be more polarized or moderated than the representative agent's peak. When the efforts of the opposite group are strategically chosen and the CSF is homogeneous of degree zero, we showed that the inter-group positive effects of moderation always offset the intra-group effects due to heterogeneity, so that the optimal target-policy of any representative is always more moderated

[24]We can interpret this setting as a moral hazard problem in the context of a public good provision with non-transferable utility where the size of the public good depends on the aggregate effort of the group members.

than its peak. This is independent of the groups' sizes and of how this moderation alters the aggregate surplus of the group. These results hold with either convex or linear costs of effort. However, when the CSF has the linear-difference form analyzed in this paper, the optimal policy can be either more moderated or polarized than the representative's peak. Polarization will arise when the representative of the challenging group is sufficiently moderate and the group defending the status-quo is sufficiently small because in this case the inter-group positive effects of moderation will not offset the intra-group forces. The paper also analyzes the case where the challenging group has not an exogenous representative and the target-policy is collectively selected. In this case, the heterogeneity among challengers imply that there is no unanimous consent on the best policy to lobby for in the subsequent contest. The results are similar to those obtained in the main setting of the paper: any Condorcet winner policy must always imply a moderation with respect to the median's peak under a CSF homogeneous of degree zero either with convex or linear costs of effort but not under the linear-difference CSF. In this case, the Condorcet winner policy that will be confronted to the opposite group policy might be more polarized than the median's peak.

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Appendix A. Proofs of results in Section 2

Proof of Proposition 1. Given $\bar{Q}(x)$ and $a_r^* = a_r(x)$ for all $r \in N$, the indirect utility function of any representative $r \in N$ can be written as,

$$V_r(x) = f\left(\bar{Q}(x)\right) D_r(x) + u_r(1) - \frac{a_r^2(x)}{2}.$$

Differentiating this function yields

$$V_r'(x) = f'\left(\bar{Q}(x)\right) D_r(x)\,\bar{Q}'(x) + f\left(\bar{Q}(x)\right) D_r'(x) - a_r(x)\,a_r'(x).$$

Using (1), we obtain

$$a_r'(x) = f''\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_r(x)\,\bar{Q}'(x) + f'\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_r'(x).$$

Thus,

$$\begin{aligned}
V_r'(x) &= f'\left(\bar{Q}(x)\right) D_r(x)\,\bar{Q}'(x) + f\left(\bar{Q}(x)\right) D_r'(x) \\
&\qquad - a_r(x)\left[f''\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_r(x)\,\bar{Q}'(x) + f'\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_r'(x)\right] \\
&= a_r(x)\,\bar{B}\bar{Q}'(x) + f\left(\bar{Q}(x)\right) D_r'(x) \\
&\qquad - a_r(x)\left[f''\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_r(x)\,\bar{Q}'(x) + f'\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_r'(x)\right] \\
&= a_r(x)\,\bar{B}\left[1 - f''\left(\bar{Q}(x)\right)\frac{1}{\bar{B}^2} D_r(x)\right]\bar{Q}'(x) \\
&\qquad + \left[f\left(\bar{Q}(x)\right) - a_r(x)\,f'\left(\bar{Q}(x)\right)\frac{1}{\bar{B}}\right] D_r'(x).
\end{aligned}$$

From the concavity of $f(Q)$ and the fact that $f(Q) \in [0,1]$, it is immediate that $f(Q) > f'(Q)\,Q$ for all $Q > 0$. Hence,

$$f\left(\bar{Q}(x)\right) - f'\left(\bar{Q}(x)\right)\frac{a_r(x)}{\bar{B}} > f\left(\bar{Q}(x)\right) - f'\left(\bar{Q}(x)\right)\bar{Q}(x) > 0.$$

Therefore, as $1 - f''\left(\bar{Q}(x)\right)\frac{1}{\bar{B}^2} D_r(x) > 0$, we obtain that any optimal solution must satisfy $\bar{Q}'(x)\,D_r'(x) < 0$. Moreover, when $r < \overline{x}$ then (i) $\bar{Q}'(x) > 0$ and $D_r'(x) \geq 0$ for all $x \leq r$ and (ii) $\bar{Q}'(x) \leq 0$ and $D_r'(x) < 0$ for all $x \geq \overline{x}$. Similarly in cases where $r > \overline{x}$. Thus, the statement of the proposition follows. □

Proof of Lemma 1. First, we show that $\left|\frac{D_N'(x)}{D_N(x)}\right| < \left|\frac{D_M'(x)}{D_M(x)}\right|$. Given that

$$\begin{aligned}
\left|\frac{D_N'(x)}{D_N(x)}\right| &= \left|\frac{\sum_{j\in N} D_j'(x)}{\sum_{j\in N} D_j(x)}\right| \leq \frac{\sum_{j\in N}\theta'(|x-j|)}{\sum_{j\in N}\theta(|1-j|) - \sum\theta(|x-j|)}, \\
\left|\frac{D_M'(x)}{D_M(x)}\right| &= \frac{m\theta'(|1-x|)}{m\theta(|1-x|)},
\end{aligned}$$

then

$$\left|\frac{D_N'(x)}{D_N(x)}\right| - \left|\frac{D_M'(x)}{D_M(x)}\right| \leq \frac{1}{K}\left\{\theta(|1-x|)\sum_{j\in N}\theta'(|x-j|)\right.$$
$$\left. -\theta'(|1-x|)\sum_{j\in N}\left[\theta(|1-j|)-\theta(|x-j|)\right]\right\},$$

where $K \equiv \left(\sum_{j\in N}\theta(|1-j|) - \sum_{j\in N}\theta(|x-j|)\right)\theta(|1-x|) > 0$. From the convexity of θ, we have that

$$\begin{aligned}
\frac{\theta(|1-j|)-\theta(|x-j|)}{1-x} &> \theta'(|x-j|) \text{ for all } j\in N, \text{ and} \\
\frac{\theta(|1-x|)}{1-x} &< \theta'(|1-x|) \text{ for all } j\in M.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \left|\frac{D_N'(x)}{D_N(x)}\right| - \left|\frac{D_M'(x)}{D_M(x)}\right| \\
< \; & \frac{1}{k}\left\{\theta(|1-x|)\sum\theta'(|x-j|) - \theta'(|1-x|)(1-x)\sum\theta'(|x-j|)\right\} \\
< \; & \frac{1}{k}\left\{\theta'(|1-x|)(1-x)\sum\theta'(|x-j|) - \theta'(|1-x|)(1-x)\sum\theta'(|x-j|)\right\} = 0.
\end{aligned}$$

Given that

$$Q'(x) = \frac{1}{2}Q^{-1}(x)\left[\frac{D_N'(x)}{D_M(x)} - \frac{D_N(x)\,D_M'(x)}{D_M^2(x)}\right] = \frac{1}{2}Q(x)\left[\frac{D_N'(x)}{D_N(x)} - \frac{D_M'(x)}{D_M(x)}\right]$$

and that $D_M'(x) < 0$, the claim follows. □

Proof of Proposition 2. Differentiating the indirect utility function of any representative r,

$$V_r(x) = f(Q(x))\,D_r(x) + u_r(1) - \frac{1}{2}a_r^2(x),$$

where $a_r(x)$ is given by (6) we obtain

$$\begin{aligned}
V_r'(x) \;=\; & f'(Q(x))\,Q'(x)\,D_r(x) + f(Q(x))\,D_r'(x) \\
& -\frac{1}{2}f''(Q(x))\,Q(x)\,Q'(x)\,\frac{D_r^2(x)}{D_N(x)} - \frac{1}{2}f'(Q(x))\,Q'(x)\,\frac{D_r^2(x)}{D_N(x)} \\
& -f'(Q(x))\,Q(x)\,\frac{D_r(x)}{D_N(x)}D_r'(x) + \frac{1}{2}f'(Q(x))\,Q(x)\,\frac{D_r^2(x)}{D_N^2(x)}D_N'(x).
\end{aligned}$$

For $x \le r$, $D_r'(x) \ge 0$. Moreover from the concavity of f, $f(Q(x)) > f'(Q(x))\,Q(x)$. Consequently,

$$f(Q(x))\,D_r'(x) - f'(Q(x))\,Q(x)\left(\frac{D_r(x)}{D_N(x)}\right)D_r'(x) \ge 0$$

Hence, using $f''(Q(x)) < 0$ and $Q'(x) > 0$ (see Lemma 1),

$$\begin{aligned} V_r'(x) &> f'(Q(x))\,Q'(x)\,D_r(x) - \frac{1}{2}f'(Q(x))\,Q'(x)\left(\frac{D_r^2(x)}{D_N(x)}\right) \\ &\quad + \frac{1}{2}f'(Q(x))\,Q(x)\left(\frac{D_r^2(x)}{D_N^2(x)}\right)D_N'(x) \\ &= f'(Q(x))\,Q'(x)\,D_r(x)\left[1 - \frac{1}{2}\left(\frac{D_r(x)}{D_N(x)}\right)\right] \\ &\quad + \frac{1}{2}f'(Q(x))\,Q(x)\left(\frac{D_r^2(x)}{D_N^2(x)}\right)D_N'(x). \end{aligned} \tag{A.1}$$

It is immediate that this derivative is positive when $D_N'(x) \ge 0$. Thus, $x_r^* > \min\{\overline{x}, r\}$, implying $x_r^* > r$ when $\overline{x} \ge r$.

To complete the proof, let consider $\overline{x} < r$ and $x \in (\overline{x}, r]$ so that $D_N'(x) < 0$ and $D_r'(x) \ge 0$.

From (A.1), and using $Q'(x) = \frac{1}{2}Q(x)\left(\frac{D_N'(x)}{D_N(x)} - \frac{D_M'(x)}{D_M(x)}\right)$ we obtain

$$\begin{aligned} V_r'(x) &> \frac{1}{2}Q(x)\left(\frac{D_N'(x)}{D_N(x)} - \frac{D_M'(x)}{D_M(x)}\right)f'(Q(x))\,D_r(x)\left[1 - \frac{1}{2}\left(\frac{D_r(x)}{D_N(x)}\right)\right] \\ &\quad + \frac{1}{2}f'(Q(x))\,Q(x)\left(\frac{D_r^2(x)}{D_N^2(x)}\right)D_N'(x). \end{aligned}$$

Thus,

$$V_r'(x) > \frac{1}{4}Q(x)\,f'(Q(x))\,\frac{D_r(x)}{D_N^2(x)\,D_M(x)}R(x),$$

where

$$R(x) = D_N'(x)\,D_M(x)\,[2D_N(x) + D_r(x)] - D_N(x)\,D_M'(x)\,[2D_N(x) - D_r(x)].$$

Under quadratic preferences,

$$\begin{aligned}
\frac{R(x)}{2mn(1-x)^3} &= -(x-\overline{x})(2n-2r+x-4n\overline{x}+2nx+1) \\
&\quad +(1+x-2\overline{x})(-x+2r+2n-4n\overline{x}+2nx-1) \\
&= (1+x-2\overline{x})(2n-4n\overline{x}+2nx-[1+x-2r]) \\
&\quad -(x-\overline{x})(2n-4n\overline{x}+2nx+[1+x-2r]) \\
&= [(1+x-2\overline{x})-(x-\overline{x})][2n-4n\overline{x}+2nx] \\
&\quad -[1+x-2r][(1+x-2\overline{x})+(x-\overline{x})] \\
&= 2n(1-\overline{x})(1+x-2\overline{x})-(1+2x-3\overline{x})(1+x-2r) \\
&> 2n(1-\overline{x})(1+x-2r)-(1+2x-3\overline{x})(1+x-2r) \\
&= [2n(1-\overline{x})-(3-3\overline{x})+2-2x](1+x-2r) \\
&= [(2n-3)(1-\overline{x})+2(1-x)](1+x-2r)>0.
\end{aligned}$$

Therefore $R(x)>0$ implying $V_r'(x)>0$ and the claim follows. □

Appendix B. Proofs of results in Section 3

Proof of Proposition 3. We start by proving two preliminary results. Let $j \in N$.

Lemma 3. *Any $x \in (0,\overline{x}) \cup (d,1)$ is majority blocked, either by $\overline{x}$ or by d.*

Proof. It follows immediately from the proof of Proposition 1 that, if $x \leq \min\{j,\overline{x}\}$ then $V_j'(x)>0$ and $V_j'(x)<0$ for all $x \geq \max\{j,\overline{x}\}$ and $j \in N$. This implies that either j or $\overline{x}$ is preferred by j to any $x \in (0,\overline{x}) \cup (j,1)$ and the claim follows from the definition of the median member, d. □

We now define,

$$\begin{aligned}
H(x,j) &= V_d(x)-V_j(x)= \\
&\qquad f\left(\bar{Q}(x)\right)[D_d(x)-D_j(x)]-\frac{1}{2}\left[\frac{f'\left(\bar{Q}(x)\right)}{\bar{B}}\right]^2\left[D_d^2(x)-D_j^2(x)\right].
\end{aligned}$$

We first prove the following lemma,

Lemma 4. *For quadratic preferences: (i) $\frac{\partial H(x,j)}{\partial x} \leq 0$ for all $x \in [\overline{x},d]$ iff $j \geq d$ and (ii) $\frac{\partial H(x,j)}{\partial x} \geq 0$ for all $x \in [d,\overline{x}]$ iff $j \leq d$.*

Proof. Differentiating $H(x,j)$ we obtain

$$\begin{aligned}\frac{\partial H(x,j)}{\partial x} &= f'\left(\bar{Q}(x)\right)[D_d(x)-D_j(x)]\,\bar{Q}_0'(x)\left\{1-\frac{f''\left(\bar{Q}(x)\right)}{B^2}[D_d(x)+D_j(x)]\right\}\\ &\quad +f\left(\bar{Q}(x)\right)\left[D_d'(x)-D_j'(x)\right]\\ &\quad -\left[\frac{f'\left(\bar{Q}(x)\right)}{B}\right]^2\left[D_d(x)\,D_d'(x)-D_j(x)\,D_j'(x)\right].\end{aligned}$$

As $f'\left(\bar{Q}\right)\bar{Q}=f'\left(\bar{Q}\right)\left(A^*/\bar{B}\right)=\left(f'\left(\bar{Q}\right)/\bar{B}\right)^2 D_N(x)$ and

$$\bar{Q}'(x)=\frac{\bar{Q}(x)}{1-f''\left(\bar{Q}(x)\right)\left(1/\bar{B}^2\right)D_N(x)}\frac{D_N'(x)}{D_N(x)},$$

the previous expression can be written as

$$\begin{aligned}\frac{\partial H(x,j)}{\partial x} &= f'\left(\bar{Q}(x)\right)\bar{Q}(x)[D_d(x)-D_j(x)]\,W(x)\frac{D_N'(x)}{D_N(x)}\\ &\quad +f\left(\bar{Q}(x)\right)\left[D_d'(x)-D_j'(x)\right]\\ &\quad -\frac{f'\left(\bar{Q}(x)\right)\bar{Q}(x)}{D_N(x)}\left[D_d(x)\,D_d'(x)-D_j(x)\,D_j'(x)\right],\end{aligned} \tag{B.1}$$

where

$$W(x)=\frac{1-\frac{f''\left(\bar{Q}(x)\right)}{\bar{B}^2}[D_d(x)+D_j(x)]}{1-\frac{f''\left(\bar{Q}(x)\right)}{\bar{B}^2}D_N(x)}\in(0,1).$$

Note also that $f\left(\bar{Q}\right)\geq f'\left(\bar{Q}\right)\bar{Q}$.

Case 1: $x\in[\overline{x},d]$ and $j>d$. As $D_d(x)>D_j(x)>0$, $0<D_d'(x)<D_j'(x)$ and $D_N'(x)<0$, we obtain

$$\begin{aligned}\frac{\partial H(x,j)}{\partial x} &< f'\left(\bar{Q}(x)\right)\bar{Q}(x)\left\{[D_d(x)-D_j(x)]\,W\frac{D_N'(x)}{D_N(x)}+\left[D_d'(x)-D_j'(x)\right]\right.\\ &\qquad \left.-\frac{D_d(x)\,D_d'(x)-D_j(x)\,D_j'(x)}{D_N(x)}\right\}\\ &< \frac{f'\left(\bar{Q}(x)\right)\bar{Q}(x)}{D_N(x)}\left\{D_N(x)\left[D_d'(x)-D_j'(x)\right]\right.\\ &\qquad \left.-\left[D_d(x)\,D_d'(x)-D_j(x)\,D_j'(x)\right]\right\}\\ &< \frac{f'\left(\bar{Q}(x)\right)\bar{Q}(x)}{D_N(x)}\left\{[D_d(x)+D_j(x)]\left[D_d'(x)-D_j'(x)\right]\right.\\ &\qquad \left.-\left[D_d(x)\,D_d'(x)-D_j(x)\,D_j'(x)\right]\right\}\\ &= \frac{f'\left(\bar{Q}(x)\right)\bar{Q}(x)}{D_N(x)}\left\{D_j(x)\,D_d'(x)-D_d(x)\,D_j'(x)\right\}<0.\end{aligned}$$

Case 2: $x \in [\overline{x}, d]$ and $j < d$. In these cases, $D_d(x) < D_j(x)$, $0 < D'_d(x)$, $D'_d(x) > D'_j(x)$ and $D'_N(x) < 0$. Thus, proceeding as before, we obtain

$$\frac{\partial H(x,j)}{\partial x} > \frac{f'\left(\bar{Q}(x)\right)\bar{Q}(x)}{D_N(x)}\left\{D_j(x)\,D'_d(x) - D_d(x)\,D'_j(x)\right\} > 0.$$

Case 3: $x \in [\overline{x}, d]$ and $j > d$. Now, $D_d(x) > D_j(x) > 0$, $D'_d(x) < 0$, $D'_d(x) < D'_j(x)$ and $D'_N(x) > 0$. Using (B.1) and $W < 1$, we get

$$\begin{aligned}\frac{\partial H(x,j)}{\partial x} &< \frac{f'\left(\bar{Q}(x)\right)\bar{Q}(x)}{D_N(x)}\left\{\left[D_d(x) - D_j(x)\right]D'_N(x) + D_N(x)\left[D'_d(x) - D'_j(x)\right]\right.\\ &\quad \left. - \left[D_d(x)\,D'_d(x) - D_j(x)\,D'_j(x)\right]\right\}.\end{aligned}$$

When preferences are quadratic, $D_j(x) = (1-x)(1+x-2j)$ and $D'_j(x) = -2(x-j)$ for all $j \in N$. Thus,

$$D_d(x) - D_j(x) = 2(1-x)(j-d)$$

and

$$D'_d(x) - D'_j(x) = -2(j-d).$$

Also, $D_N(x) = n(1-x)(1+x-2\overline{x})$ and $D'_N(x) = -2n(x-\overline{x}) > 0$. Substituting the values for quadratic preferences, we obtain

$$\frac{\partial H(x,j)}{\partial x} < \frac{f'\left(\bar{Q}(x)\right)\bar{Q}(x)}{D_N(x)}\left\{-2(1-x)(j-d)(2d-3x-1-4n\overline{x}+3nx+n+2j)\right\}.$$

Since $\overline{x} \leq \frac{1}{n}\left(\frac{n+1}{2}d + j + \frac{n-3}{2}\frac{1}{2}\right)$, $x \in [d, \overline{x}]$ and $j > d$, it can be concluded that $\frac{\partial H(x,j)}{\partial x} < 0$.

Case 4: Proceeding as in Case 3, if $j < d$ then $D_d(x) - D_j(x) < 0$ and $D'_d(x) - D'_j(x) > 0$. Thus,

$$\frac{\partial H(x,j)}{\partial x} > \frac{f'\left(\bar{Q}(x)\right)\bar{Q}(x)}{D_N(x)}\left\{-2(1-x)(j-d)(2d-3x-1-4n\overline{x}+3nx+n+2j)\right\}.$$

Since $(2d-3x-1-4n\overline{x}+3nx+n+2j) > 0$ and $(j-d) < 0$ then $\frac{\partial H(x,j)}{\partial x} > 0$. $\square$

To complete the proof, we next show that the optimal target-policy of the median player beats any other alternative. We consider the case where $\overline{x} < d$; a similar argument will prove the statement for cases where $\overline{x} > d$.

By Lemma 3, any $x \in (0, \overline{x}) \cup (d, 1)$ is majority blocked, either by $\overline{x}$ or by d. By Lemma 4, for any $x, y \in [\overline{x}, d]$, we have that

1. If $V_d(x) \geq V_d(y)$ for some $x > y \Longrightarrow V_j(x) > V_j(y)$ for all $j > d$
2. If $V_d(x) \geq V_d(y)$ for some $x < y \Longrightarrow V_j(x) > V_j(y)$ for all $j < d$.

Thus, no other $y \neq x_d^*$ such that $y \in [\overline{x}, d]$, gets the support of a majority, as preferences satisfy single-crossing on the interval $[\overline{x}, d]$. The only Condorcet-winner candidate is x_d^*, as (i) it is majority preferred to any $y \in [\overline{x}, d]$ and (ii) either $\overline{x}$ or d majority block any other $y \in (0, \overline{x}) \cup (d, 1)$. □

Proof of Proposition 4. We start by proving a preliminary result.

Lemma 5. *For any three agents $i, j, k \in N$ such that $i < j < k$ and two policies $x, y \in X$, if $V_j(x) > V_j(y)$, $V_i(x) < V_i(y)$ and $V_k(x) < V_k(y)$ then $y > x$.*

Proof. The three inequalities can be written as

$$\begin{aligned}
f(Q(x)) D_j(x) - \frac{1}{2} a_j^2(x) &> f(Q(y)) D_j(y) - \frac{1}{2} a_j^2(y), \\
f(Q(x)) D_k(x) - \frac{1}{2} a_k^2(x) &< f(Q(y)) D_k(y) - \frac{1}{2} a_k^2(y), \\
f(Q(x)) D_i(x) - \frac{1}{2} a_i^2(x) &< f(Q(y)) D_i(y) - \frac{1}{2} a_i^2(y).
\end{aligned}$$

Since $a_l^2(x) = \left[\frac{f'(Q(x))}{B(x)}\right]^2 D_l^2(x)$ for any $l \in N$ we have that

$$\begin{aligned}
& [D_j(x) - D_i(x)] \left\{ f(Q(x)) - \frac{1}{2} \left[\frac{f'(Q(x))}{B(x)}\right]^2 [D_j(x) + D_i(x)] \right\} \\
> \; & [D_j(y) - D_i(y)] \left\{ f(Q(y)) - \frac{1}{2} \left[\frac{f'(Q(y))}{B(y)}\right]^2 [D_j(y) + D_i(y)] \right\}.
\end{aligned}$$

Given that $D_l(x) - D_t(x) = 2(1-x)(t-l)$ for any $l, t \in N$ we get

$$\begin{aligned}
& (1-x) \left\{ f(Q(x)) - \frac{1}{2} \left[\frac{f'(Q(x))}{B(x)}\right]^2 [D_j(x) + D_k(x)] \right\} \\
> \; & (1-y) \left\{ f(Q(y)) - \frac{1}{2} \left[\frac{f'(Q(y))}{B(y)}\right]^2 [D_j(y) + D_k(y)] \right\}
\end{aligned}$$

and

$$\begin{aligned}
& (1-x) \left\{ f(Q(x)) - \frac{1}{2} \left[\frac{f'(Q(x))}{B(x)}\right]^2 [D_j(x) + D_i(x)] \right\} \\
< \; & (1-y) \left\{ f(Q(y)) - \frac{1}{2} \left[\frac{f'(Q(y))}{B(y)}\right]^2 [D_j(y) + D_i(y)] \right\}.
\end{aligned}$$

Hence subtracting the previous inequalities,

$$(1-x)\left[\frac{f'(Q(x))}{B(x)}\right]^2\{D_i(x)-D_k(x)\}>(1-y)\left[\frac{f'(Q(y))}{B(y)}\right]^2\{D_i(y)-D_k(y)\}.$$

Substituting $B^2(x)=f'(Q(x))\,Q(x)\,D_M(x)$ and $D_M(x)=m(1-x)^2$ we get

$$\left[\frac{f'(Q(x))}{Q(x)(1-x)}\right]\{D_i(x)-D_k(x)\}>\left[\frac{f'(Q(y))}{Q(y)(1-y)}\right]\{D_i(y)-D_k(y)\}.$$

Replacing $D_i(x)-D_k(x)=2(1-x)(k-i)$, we have that

$$\frac{f'(Q(x))}{Q(x)}>\frac{f'(Q(y))}{Q(y)},$$

implying $Q(x)<Q(y)\Longleftrightarrow x<y$. □

The previous result implies that if agents i and j, such that $i<d<j$, prefer a policy x to x_d^* then $x\geq x_d^*$. Then, if a Condorcet-winner policy exists, it can not be more polarized than x_d^*. □

Appendix C. Intermediate results in Section 4.1

In the NSO case (where $B(x)=\bar{B}$), implicit differentiation of (10) yields

$$\bar{Q}'(x)=\frac{A'(x)}{\bar{B}}=-\frac{f'\left(A/\bar{B}\right)D_0'(x)}{f''\left(A/\bar{B}\right)D_0(x)}<0 \text{ for all } x>0.$$

Hence, the optimal target-policy of any representative $r\in N$, $r\neq 0$, solves

$$\begin{aligned} 0 &= f'(Q(x))\,Q'(x)\,D_r(x)+f(Q(x))\,D_r'(x)\\ &= D_r(x)\left[f(Q(x))\frac{D_r'(x)}{D_r(x)}-\frac{\left(f'\left(A/\bar{B}\right)\right)^2}{f''\left(A/\bar{B}\right)}\frac{D_0'(x)}{D_0(x)}\right], \end{aligned}$$

implying $D_0'(x)\cdot D_r'(x)<0$. That is, $x_r^*\in(0,r)$ for all $r\neq 0$. Regarding agent 0, her optimal target-policy is $x_0^*=0$ as $V_0'(x)<0$ for all $x>0$.

Lemma 6. *If $x<z$ and $i<j$ then*

$$\frac{D_i(z)}{D_i(x)}<\frac{D_j(z)}{D_j(x)}. \tag{C.1}$$

Proof. To prove the statement we consider six possible cases depending on the relative position of x, z, i, j. In each case, we prove that: if $x < z$ and $i < j$ then $D_j(z) - D_j(x) > D_i(z) - D_i(x)$. Therefore, as $D_j(x) < D_i(x)$, it follows that

$$\frac{D_j(z) - D_j(x)}{D_j(x)} > \frac{D_i(z) - D_i(x)}{D_i(x)},$$

which implies (C.1).

Note that $D_k(z) - D_k(x) = \theta(|k - x|) - \theta(|k - z|)$, where θ is strictly convex.

Case 1. $x < z \leq i < j$. In this case, we have that

$$\theta(|j - x|) - \theta(|j - z|) > \theta(|i - x|) - \theta(|i - z|).$$

So, the statement follows.

Case 2. $i < j \leq x < z$. Proceeding as in the previous case, we obtain

$$\theta(|z - i|) - \theta(|x - i|) > \theta(|z - j|) - \theta(|x - j|).$$

Case 3. $x \leq i < j \leq z$. In these cases

$$\begin{aligned} \theta(|j - x|) &> \theta(|j - i|) + \theta(|i - x|) \quad \textit{and} \\ \theta(|z - i|) &> \theta(|j - i|) + \theta(|z - j|). \end{aligned}$$

Hence, $\theta(|j - x|) - \theta(|j - z|) > \theta(|i - x|) + \theta(|z - i|)$.

Case 4. $x \leq i \leq z \leq j$. The statement follows directly, as the convexity of θ implies

$$\theta(|j - x|) > \theta(|j - z|) + \theta(|z - i|) + \theta(|i - x|) > \theta(|j - z|) + \theta(|i - x|) - \theta(|z - i|).$$

Case 5. $i \leq x \leq j < z$. Now, as before we obtain

$$\theta(|z - i|) > \theta(|x - i|) + \theta(|j - x|) + \theta(|z - j|) > \theta(|x - i|) - \theta(|j - x|) + \theta(|z - j|).$$

Case 6. $i \leq x \leq z \leq j$. In these cases,

$$\theta(j - x) - \theta(j - z) > 0 > \theta(x - i) - \theta(z - i).$$

$\square$

Lemma 7. *For $z > x$,*

$$V_d(z) \geq V_d(x) \Longrightarrow V_l(z) > V_l(x) \ \textit{for all } l > d \neq 0.$$

Proof. Let us assume that $V_d(z) - V_d(x) \geq 0$, *i.e.* $f(Q(z)) D_d(z) - f(Q(x)) D_d(x) \geq 0$. Then, from Lemma 6, it can be said that for all $l > d$

$$\frac{f(Q(x))}{f(Q(z))} \leq \frac{D_d(z)}{D_d(x)} < \frac{D_l(z)}{D_l(x)}.$$

Hence,

$$V_l(x) = f(Q(x)) D_l(x) < f(Q(z)) D_l(z) = V_l(z).$$

$\square$

Appendix D. Proof of results in Section 4.2

Proof of Proposition 5. Since $D'_N = 2n(\overline{x} - x)$ and $D'_M = 2m(x-1)$,

$$P' = 2s^2 (n\overline{x} - x(n+m) + m).$$

So, $P' > 0 \Leftrightarrow x < \hat{x}$ and $P' < 0 \Leftrightarrow x > \hat{x}$.

Notice that $V'_j(x) = P'D_j + D'_j(P - s^2 D_j)$. Under quadratic preferences $D_j(x) = (1-x)(1+x-2j)$ and $D'_j(x) = 2(j-x)$, so $V'_j(j) = P'D_j(j)$. Since $D_j(j) > 0$ for any $j < 1/2$ and $x < 1$, the statement of the proposition follows. $\square$

Proof of Lemma 2. The indirect utility function of any agent $j \in N$ is

$$V_j(x) = (1-x)(1+x-2j)\, G \geq 0,$$

where

$$G = \frac{1}{2}\left[1 - s^2(1-x)(1+x-2j)\right] + s^2(1-x)(n(1+x-2\overline{x}) - m(1-x)).$$

Since $j \leq 1/2$ and $x \in [0,1]$, $G \geq 0$ and $V_j \geq 0$. Thus,

$$V'_j(x) = 2(j-x)\,G + (1-x)(1+x-2j)\frac{\partial G}{\partial x}.$$

Note that, as $j \leq 1/2$, at any interior optimum $(j-x)\frac{\partial G}{\partial x} \leq 0$, where

$$\frac{\partial G}{\partial x} = -s^2(j - 2m - x - 2\overline{x}n + 2mx + 2nx).$$

The second partial derivative is

$$\begin{aligned} V''_j(x) &= -2G + 4(j-x)\frac{\partial G}{\partial x} + (1-x)(1+x-2j)\frac{\partial^2 G}{\partial x^2} \\ &= -2G + 4(j-x)\frac{\partial G}{\partial x} - s^2(2m+2n-1)(1-x)(1+x-2j), \end{aligned}$$

which is negative when $x \in [0,1]$. $\square$

Proof of Proposition 6. Lemma 2 guarantees the existence of a Condorcet winner policy x_w^*. We start by proving an intermediate result.

Lemma 8. *For any $x \in [0,1]$ and any two agents $i, k \in N$ such that $i < k$, $V_i'(x) < 0$ and $V_k'(x) < 0$ imply $V_j'(x) < 0$ for any $j \in (i,k)$.*

Proof. Considering quadratic preferences, for any $l \in N$, V_l' can be written as

$$V_l'(x) = \Phi l^2 + \Psi l + \Omega,$$

where Φ, Ψ and Ω are expressions that depend on $s, x, \overline{x}, m$ and n. Let $j = \alpha i + (1-\alpha) k$ and assume $\alpha V_i'(x) < 0$ and $(1-\alpha) V_k'(x) < 0$.

Hence,

$$\begin{aligned} & \alpha\Phi i^2 + \alpha\Psi i + \alpha\Omega + (1-\alpha)\Phi k^2 + (1-\alpha)\Psi k + (1-\alpha)\Omega \\ = \; & \alpha\Phi i^2 + (1-\alpha)\Phi k^2 + \Psi j + \Omega < 0. \end{aligned} \tag{D.1}$$

On the other hand,

$$V_j'(x) = \Phi(\alpha i + (1-\alpha) k)^2 + \Psi j + \Omega.$$

Using (D.1), we obtain

$$V_j'(x) < \Phi(\alpha i + (1-\alpha) k)^2 - \alpha\Phi i^2 - (1-\alpha)\Phi k^2 = -\Phi\alpha(1-\alpha)(k-i)^2 < 0.$$

□

This implies that $x_w^* \geq x_d^*$. Therefore, by Proposition 5 we can conclude that if $d < \hat{x}$ then $x_w^* \geq d$. Now, it remains to be shown that if $d > \hat{x}$ then $x_w^* \leq d$. By contradiction let us assume that $x_w^* > d$. By Lemma 8 this can only happen when

$$V_0'(d) = P'D_0 + D_0'\left(P - s^2 D_0\right) > 0,$$

where 0 denotes the agent whose peak is at zero. Given that $D_0'(d) < 0$ and $P'(d) < 0$ (since P is maximized at $\hat{x}$ and $d > \hat{x}$), necessarily $P(d) - s^2 D_0(d) < 0$. Under quadratic preferences this can be written as

$$\frac{1}{2} + s^2(1-d)(n(1+d-2\overline{x}) - (1+d) - m(1-d)) < 0$$

implying

$$\frac{1}{2} + s^2(n(1+d-2\overline{x}) - (1+d) - m(1-d)) < 0.$$

Additionally, the existence of an interior equilibrium for any x and in particular for $x = 0$ requires that

$$P(0) = \frac{1}{2} + s^2(n(1-2\overline{x}) - m) \geq 0.$$

The last two inequalities can both hold only when $d \leq 1/(n+m+1)$. Since $d > \hat{x}$ and $\hat{x} > 1/(n+m+1)$ we have the contradiction that concludes the proof. □

Appendix E. Results in the model with the linear-difference CSF and a NSO.

In the NSO case (where $B(x) = \bar{B}$), the following is obtained:

Proposition 7. *(NSO case) Under the linear-difference CSF, either* $(x_r^* - r)(x_r^* - \overline{x}) < 0$ *or* $x_r^* = \overline{x} = r$*, for any* $r \in N$.

Proof of Proposition 7. Differentiating the indirect utility function of agent j yields

$$V_j'(x) = P'(x)\, D_j(x) + \left(P(x) - s^2 D_j(x)\right) D_j'(x).$$

Additionally, the existence of an interior equilibrium for any x and in particular for $x = 1$ (the case that implies a lower P) requires that $P(1) = \frac{1}{2} - s\bar{B} \geq 0$. Given that $D_{-j}(x) = \sum_{i \in N, i \neq j} D_i(x) \geq 0$, this implies that

$$P - s^2 D_j = \frac{1}{2} + s\left(sD_N - \bar{B}\right) - s^2 D_j = \frac{1}{2} + s\left(sD_{-j} - \bar{B}\right) \geq 0.$$

Consequently, $V_j'(x) = 0$ implies that $P'D_j' < 0$ in equilibrium, where $P' = s^2 D_N'$. Since D_j and D_N are uniquely maximal at j and $\overline{x}$, respectively, the statement of the proposition follows.[25] □

As in the case of a CSF homogeneous of degree zero (see Proposition 1), this result comes from the interaction of intra-group forces that lead the representative to face the trade-off between the winning-probability maximizing policy ($\overline{x}$) and the winning-utility maximizing policy (r). Consequently, the optimal target-policy of the representative agent $r \in N$ can imply either a moderation or a polarization with respect to her peak depending on whether $r < \overline{x}$ or $r > \overline{x}$, respectively.

If the policy is collectively selected, the following result guarantees the existence of a Condorcet winner policy.

Lemma 9. *(NSO case) Under quadratic preferences and the linear-difference CSF, the indirect utility is single-peaked.*

Proof of Lemma 9. The indirect utility function can be written as, $V_j(x) = D_j G + u_j(1)$, where $G = P - \frac{s^2}{2} D_j \geq 0$ and $D_j \geq 0$. Thus,

$$V_j'(x) = D_j' G + D_j \frac{\partial G}{\partial x}.$$

At any interior optimum we must have,

$$D_j' \frac{\partial G}{\partial x} \leq 0,$$

[25] Notice that the proof is valid not only for quadratic preferences.

where $D'_j \frac{\partial G}{\partial x} = 2s^2(j-x)(D'_N - \frac{1}{2}D'_j) = 2s^2(j-x)(2n(\overline{x}-x)-(j-x))$. The second partial derivative is,

$$V''_j(x) = D''_j G + 2D'_j \frac{\partial G}{\partial x} + \frac{\partial^2 G}{\partial x^2},$$

where $\frac{\partial^2 G}{\partial x^2} = -s^2(2n-1) \leq 0$, as $D''_j = -2 < 0$, and we know that at any interior optimum $D'_j \frac{\partial G}{\partial x} \leq 0$, then the interior optimum is actually a maximum. □

The following result shows the relative location of this Condorcet-winner policy x^*_w.

Proposition 8. *(NSO case) Under quadratic preferences and the linear-difference CSF, if* $d \neq \overline{x}$ *then* $(x^*_w - d)(x^*_w - \overline{x}) < 0$. *Otherwise,* $x^*_w = d = \overline{x}$.

Proof of Proposition 8. Lemma 9 guarantees the existence of a Condorcet winner policy. Let us consider that $d < \overline{x}$. By Proposition 7, $x^*_j > \overline{x}$ for any $j > \overline{x}$ and $x^*_j > d$ for any $j \in (d, \overline{x})$. Therefore, x^*_w cannot be lower than or equal to d. A symmetric argument proves that $x^*_w < d$ when $d > \overline{x}$. □

Consequently, unlike the case with a CSF homogeneous of degree zero, we cannot say in this case that the optimal target-policy of the median player will be the Condorcet winner. But the conclusion does not change qualitatively: the Condorcet winner optimal policy will be more moderated (or polarized) than the median's peak when this peak is lower (or higher) than $\overline{x}$.